impa Instituto de Matemática Pura e Aplicada

Hodge cycles and Gauss' hypergeometric function

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To my life partner Adrys,
to my brothers Andres and Tiago,
to my sister Andreina,
to my father Tavito,
and especially to my mother La Gordis.

Hay días en que somos tan móviles, tan móviles,
como las leves briznas al viento y al azar.
Tal vez bajo otro cielo la Gloria nos sonríe.
La vida es clara, undívaga, y abierta como un mar.

Y hay días en que somos tan fértiles, tan fértiles,
como en abril el campo, que tiembla de pasión:
bajo el influjo próvido de espirituales lluvias,
el alma está brotando florestas de ilusión.

Y hay días en que somos tan sórdidos, tan sórdidos,
como la entraña obscura de oscuro pedernal:
la noche nos sorprende, con sus profusas lámparas,
en rútiles monedas tasando el Bien y el Mal.

Y hay días en que somos tan plácidos, tan plácidos...
(¡niñez en el crepúsculo! ¡Lagunas de zafir!)
que un verso, un trino, un monte, un pájaro que cruza,
y hasta las propias penas nos hacen sonreír.

Y hay días en que somos tan lúbricos, tan lúbricos,
que nos depara en vano su carne la mujer:
tras de ceñir un talle y acariciar un seno,
la redondez de un fruto nos vuelve a estremecer.

Y hay días en que somos tan lúgubres, tan lúgubres,
como en las noches lúgubres el llanto del pinar.
El alma gime entonces bajo el dolor del mundo,
y acaso ni Dios mismo nos puede consolar.

Mas hay también ¡Oh Tierra! un día... un día... un día...
en que levamos anclas para jamás volver...
Un día en que discurren vientos ineluctables
¡un día en que ya nadie nos puede retener!

Canción de la vida profunda,
Porfirio Barba Jacob

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Frequently used notations

$(a)_n$	Pochhammer symbol defined by $(a)_n := a(a+1)\cdots(a+n-1)$.	83
β	Multi indexes, usually $\beta = (\beta', \beta_{n+1}) \in I$ with $\beta' = (\beta_1, \dots, \beta_n)$.	57
$\beta + (0, k)$	Multi indexes given by $\beta + (0', k) = (\beta_1, \dots, \beta_n, \beta_{n+1} + k)$.	78
$\delta_1 * \delta_2$	Joint cycle.	32
$\frac{\omega}{df}$	Gelfand-Leray form.	33
$\mathrm{GHod}_n(X, \mathbb{Q})_0$	Space of generic Hodge cycles.	55
ω_β	Form on $\mathbb{C}^{n+1}$ defined as $\omega_\beta = x^\beta dx := x_1^{\beta_1} \dots x_{n+1}^{\beta_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1}$.	28
$\mathrm{SHod}_n(X, \mathbb{Q})_0$	Space of strong generic Hodge cycles.	57
A_β	Rational number that allows distinguishing forms in a filtration. It is defined by $A_\beta := \sum_{j=1}^{n+1} (\beta_j + 1) \frac{v_j}{d}$. We will usually rewrite it as $A_\beta := \sum_{j=1}^{n+1} \frac{\beta_j + 1}{m_j}$.	28
$B(a_1, \cdots, a_{n+1})$	Multi parameter version of beta function.	40
$E(a, b, c)$	Gauss hypergeometric differential equation.	83
$F(a, b, c; z)$	Hypergeometric function.	83
I, J	Sets of multi indexes with $I = J \times I_m$, $J = I_{m_1} \times \cdots \times I_{m_n}$ and $I_{m_j} = \{0, 1, 2, \dots, m_j - 2\}$.	56
res	The residue map.	24

Introduction

Research Framework

The present work is devoted to the study of hypersurfaces with hypergeometric periods. We focus on a particular class, Fermat varieties perturbed by $P(y) = y(1-y)(\lambda - y)$. Periods (roughly speaking multiple integrals) are an essential part of Hodge theory that have their deepest origins in elliptic and abelian integrals. We do not aim to verify the Hodge conjecture in our examples, rather we would like to analyze transcendental properties of integration over Hodge cycles.

Deligne in 1982 explored periods of algebraic cycles. He proved that up to some constant power of $2\pi\sqrt{-1}$, the periods of algebraic cycles are algebraic with respect to the field of definition of the variety (see [Del82]). This would be also true for Hodge cycles if the Hodge conjecture holds true. In fact, Deligne proved that this property is satisfied by periods of Hodge cycles in classic Fermat varieties even though Hodge conjecture is unknown in this case. With this, he obtained algebraic relations between the values of the Γ-function on rational points. This same idea was elaborated in 2006 by Reiter and Movasati with the family

$$M_t : \quad f(x) := x_1^3 + x_2^3 + \cdots + x_5^3 - x_1 - x_2 = t,$$

to obtain algebraic relations of values of the hypergeometric functions (see [MR06]). For example, they proved that

$$e^{-\frac{5}{6}\pi i} \frac{F\left(\frac{5}{6}, \frac{1}{6}, 1; \frac{27}{16}t^2\right)}{F\left(\frac{5}{6}, \frac{1}{6}, 1; 1 - \frac{27}{16}t^2\right)}$$

belongs to $\mathbb{Q}(\zeta_3)$ for some $t \in \overline{\mathbb{Q}}$ if and only if

$$\pi^2 \frac{F\left(\frac{5}{6}, \frac{1}{6}, 1; \frac{27}{16}t^2\right)}{\Gamma\left(\frac{1}{3}\right)^3}, \quad \pi^2 \frac{F\left(\frac{5}{6}, \frac{1}{6}, 1; 1 - \frac{27}{16}t^2\right)}{\Gamma\left(\frac{1}{3}\right)^3} \in \overline{\mathbb{Q}}.$$

For instance, the above is satisfied if t is any root of the following equations

$$91125t^4 - 54000t^2 + 256, \quad 81000t^4 - 48000t^2 - 1,$$

see [MR06] and the references therein.

In this thesis, we elaborate these same ideas with the family

$$M_\lambda: \quad f(x) := x_1^{m_1} + \cdots + x_n^{m_n} + y(y-1)(y-\lambda) = 0.$$

We compute its periods and use them to give algebraic values of some hypergeometric functions (see equation (1)). This result is framed in Schwarz' work. Schwarz in [Sch73] was the first to classify hypergeometric functions which are algebraic over $\mathbb{C}(z)$. A crucial idea was to relate the hypergeometric equation with the Monodromy group. In the famous **Schwarz' list**, Schwarz determines explicit criteria for the parameters of the irreducible hypergeometric equations such that the solutions are algebraic. In the same work, Schwarz also obtained a similar but not so famous criterion in the case of reducible hypergeometric equations (see also [Kim69]).

A more general question raised by Wolfart in [Sar07] is to determine the transcendence degree of the field extension of $\overline{\mathbb{C}(z)}$ generated by the hypergeometric functions $F(a,b,c;z)$ where a, b, $c \in \mathbb{Q}$ with some fixed denominator. Or even better to determine a complete list of algebraic dependence equations among these $F(a,b,c;z)$ over the field $\overline{\mathbb{C}(z)}$. Examples of such relations are Propositions 4.8 and 4.9, Schwarz' list and Gauss' relations between contiguous hypergeometric functions. Up to the author's knowledge, Wolfart's problem remains open and without significant progress.

Main Results

Let n be an even number, $g(x) := x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$, and let P be a degree $m = m_{n+1}$ polynomial. Consider

$$f = g(x) + P(y),$$

and let F be its quasi-homogenization inside the weighted projective space $\mathbb{P}^{(1,v)}$ where $v_j = \frac{\mathrm{lcm}(m_1,\ldots,m_{n+1})}{m_j}$ for $j = 1, \ldots, n+1$. Let X be a desingularization of the weighted hypersurface $D := \{F = 0\} \subset \mathbb{P}^{(1,v)}$. We are interested in Hodge cycles of X supported in the affine part $U := \{f = 0\}$. For this, we consider a parametric family. Let

$$T := \left\{ t = (t_0, \ldots, t_m) \in \mathbb{C}^{m+1} \middle| t_m = 1, \;\; \Delta(P_t) \neq 0 \text{ where } P_t := \sum_{i=0}^{m} t_i y^i \right\}$$

be the space of polynomials of degree m with nonzero discriminant, and let

$$\mathcal{U} := \{(x,y,t) \in \mathbb{C}^n \times \mathbb{C} \times T | \; f_t(x,y) := g(x) + P_t(y) = 0\}$$

be the family of affine varieties parameterized by T. Thus, the projection $\pi : \mathcal{U} \longrightarrow T$ is a locally trivial C^∞ fibration (see [Mov20, §7.4] and the references therein). We denote by $U_t := \pi^{-1}(t) = \{f_t = 0\} \subset \mathbb{C}^{n+1}$ and X_t be a desingularization of $D_t := \{F_t = 0\} \subset \mathbb{P}^{(1,v)}$ where F_t is the quasi-homogenization of f_t.

We say that a cycle $\delta_{t_0} \in H_n(U_{t_0}, \mathbb{Q})$ is a **generic Hodge cycle** if all perturbations δ_t of it in the family T are Hodge cycles (see Definition 4.1). This space is denoted by $\mathrm{GHod}_n(X_{t_0}, \mathbb{Q})_0$. We consider a subspace of the generic Hodge cycles space by imposing certain conditions, which we call the space of **strong generic Hodge cycles** and we denote it by $\mathrm{SHod}_n(X_{t_0}, \mathbb{Q})_0$ (see Definition 4.2).

The first result of this thesis is an upper bound of the dimension of the space of strong generic Hodge cycles in certain cases (see Theorem 4.2).

Theorem 0.1. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$ and P is a polynomial of degree m.

i. For $m_1 = \cdots = m_{n-1} = 2$ and $m \geq 7$

$$\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq \begin{cases} m-1 & m_n \text{ even}, \\ 0 & m_n \text{ odd}. \end{cases}$$

ii. For $m_1 = \cdots = m_{n-2} = 2$, m_{n-1} prime, $\gcd(m_{n-1}, m_n) = 1$ and $\frac{1}{m_{n-1}} + \frac{1}{m} < \frac{1}{2}$, we have $\mathrm{SHod}_n(X, \mathbb{Q})_0 = 0$.

iii. For m_j different prime numbers, we have $\mathrm{SHod}_n(X, \mathbb{Q})_0 = 0$.

In fact, the proof of the previous theorem provides a method to calculate a set of generators of $\mathrm{SHod}_n(X, \mathbb{Q})_0$ even if $m < 7$ in the first case and if $\frac{1}{m_{n-1}} + \frac{1}{m} \geq \frac{1}{2}$ for the second case (see Corollary 4.2). Using this method, we get for example:

Corollary 0.1. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ and $P(y)$ is a polynomial of degree $m = 2, \ldots, 6$. Then

$$\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq (m-1) \left(\sum_{\substack{2 \leq d \leq \lfloor \frac{2m}{m-2} \rfloor \\ d | m_n}} \varphi(d) \right),$$

where φ is the Euler's totient function. When $m = 2$ means that $2 \leq d \leq m_n$ and $d | m_n$. Therefore for $m = 2$, $\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq (m_n - 1)$.

We obtain algebraic values of the hypergeometric function by a different method than that used by Schwarz (see Corollary 4.6). For this, we restrict ourselves to the case $P_\lambda(y) = y(1-y)(\lambda - y)$, and we compute the periods on explicit strong generic Hodge cycles. For example, we get

$$F\left(\frac{5}{6}, \frac{1}{6}, \frac{5}{3}; 1-\lambda\right), \quad F\left(\frac{7}{6}, \frac{-1}{6}, \frac{7}{3}; 1-\lambda\right) \in \overline{\mathbb{Q}(\lambda)}. \tag{1}$$

The above is somewhat exceptional given that periods are usually transcendental. Other by-products of this work are examples of non-algebraic hypergeometric functions that satisfy algebraic relations between them (see Propositions 4.8 and 4.9).

Proposition 0.1. The following expressions are in $\overline{\mathbb{Q}(\lambda)}$:

$$0 \neq 6F\left(\frac{4}{3}, -\frac{4}{3}, \frac{8}{3}; 1-\lambda\right)(\lambda^2 - \lambda + 1) - \frac{2}{3}F\left(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; 1-\lambda\right)(\lambda+1)(5\lambda^2 - 8\lambda + 5),$$

$$0 \neq 2F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)-\frac{2}{3}F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};1-\lambda\right)(\lambda+1),$$

$$\begin{aligned}0 \neq & 4F\left(\frac{2}{3},-\frac{5}{3},\frac{4}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{1}{3}F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)(\lambda+1)(8\lambda^2-11\lambda+8)+\\ & F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};1-\lambda\right)\lambda(1-\lambda)^2,\end{aligned}$$

$$\begin{aligned}0 \neq & 6F\left(\frac{2}{3},-\frac{8}{3},\frac{4}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{2}{3}F\left(\frac{2}{3},-\frac{5}{3},\frac{4}{3};1-\lambda\right)(\lambda+1)(7\lambda^2-10\lambda+7)+\\ & 2F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)\lambda(1-\lambda)^2,\end{aligned}$$

but each hypergeometric function in the expressions above is not algebraic over $\mathbb{Q}(\lambda)$. For a numerical verification of this proposition see §4.5.

We can find the algebraic functions of the expressions in Proposition 0.1 using hypergeometric theory via Gauss' relations, see Remark 4.7. Proposition 0.1 suggests that the Hodge cycles in Theorem 4.2 and Corollary 4.2 should be absolute in the sense of Deligne, see [Del82]. Moreover, the algebraic functions in Proposition 4.8 might be used in order to construct the underlying algebraic cycles explicitly, see [MS20].

Similarly to Schwarz' work, Beukers and Heckman classified the generalised hypergeometric functions which are algebraic over $\mathbb{C}(z)$ in [BH89]. On the other hand, meantime this article was being written, Movasati was able to obtain similar algebraicity properties of periods which are gathered in [Mov20, §16.9]. Apparently these periods must be related in some way to the generalised hypergeometric functions described in [BH89], for instance via a pull-back. For the classification scheme of pull-back transformations between Gauss hypergeometric differential equations see [Vid09].

Content description

This thesis is organized in the following manner.

In Chapter 1, we present Griffiths-Steenbrink's theory on the cohomology of weighted hypersurfaces. This allows us to construct a basis for the de Rham cohomology of weighted hypersurfaces. What was developed in this chapter justifies our definition of the Hodge cycle. In this chapter, we set up notation and terminology.

In Chapter 2, we present a technique to calculate periods. For this, we describe a particular basis of homology, namely joint cycles and we give a formula for the integral of differential form with pole of order one on these cycles. In this chapter we also give a basis for the homology and cohomology of affine Fermat varieties and we compute the integral on these bases, which naturally arise in the calculation of periods on a perturbation of a Fermat variety.

In Chapter 3, we explain how to reduce the order of the pole of a differential form, in order to use the formula given in the previous chapter. With this, we calculate the periods on a perturbation of a

Fermat variety, more specifically the perturbation given by $P(y) = y(1-y)(\lambda - y)$. We also provide a criterion for when a differential form in the affine part actually comes from a differential form on the compactification, this is important to establish that certain hypergeometric periods are algebraic.

In Chapter 4, we introduce the concept of strong generic Hodge cycle and we offer a method to calculate a set of generators in certain cases. With this and with what was developed in the previous chapter we find expressions involving the hypergeometric function such that they are algebraic.

In our context, the hypergeometric function appears naturally when calculating periods. That is why the appendix A, contains a review of some of the standard facts on the hypergeometric function including numerous equalities satisfied by it as well as theorems that characterize the algebraicity of this special function and which were used in the text.

CHAPTER 1

Cohomology of hypersurfaces

In this chapter we will explain how to construct a basis for the de Rham cohomology of a weighted hypersurface. The homogeneous case was developed by Griffiths in [Gri69] and the generalization to weighted hypersurfaces is due to Steenbrink [Ste77]. In §1.1 we give a brief exposition of this construction in the homogeneous case. In §1.2 we deal with weighted case. This basis will be used to define Hodge cycles on a perturbation of a Fermat variety in a explicit way as well as to compute these periods. Section §1.3 is devoted to describing Hodge cycles with affine supported via cycles at infinity. This approach was pointed out by Hossein in [Mov20]. All this will justify the definition of Hodge cycles in our case of interest given at the end of this section.

1.1 Homogeneous case

The algebraic de Rham cohomology for any smooth algebraic variety was defined by Grothendieck in [Gro66] inspired by the work of Atiyah and Hodge [HA55]. A remarkable fact is that the algebraic de Rham cohomology of a smooth projective variety X is isomorphic to the classical de Rham cohomology of the underlying C^∞ manifold X^∞. Good references for this are [Vil19] and [MV20]. That is why we will use these cohomologies interchangeably.

Let $M \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree d and $V = \mathbb{P}^{n+1} \backslash M$. By the Lefschetz hyperplane section theorem

$$H^k_{dR}(M) = \begin{cases} \mathbb{C}[\theta^{k/2}] & k \text{ even}; k \neq n \\ 0 & k \text{ odd}; k \neq n \\ \mathbb{C}^\mu & k = n \end{cases}$$

with θ the 2-form associated with the Kähler structure, also called the polarization. Therefore, the only non-trivial Hodge cycles of M lie in $H^n_{dR}(M)$. Thus, we are just interested in determining a basis for the non-trivial cohomology of M. Remember that the primitive cohomology corresponds to

$$H^n_{dR}(M)_0 := \{\omega \in H^n_{dR}(M) \big| \omega \wedge \theta = 0\}.$$

To construct the basis for $H^n_{dR}(M)$ it is enough construct the basis for $H^n_{dR}(M)_0$ because by Lefschetz decomposition we have

$$H^n_{dR}(M) = H^n_{dR}(M)_0 \bigoplus \theta^{\frac{n}{2}}\mathbb{C},$$

with θ the polarization. For this purpose we will give the generators for $H^{n+1}_{dR}(V)$ compatible with the Hodge filtration $F^{k+1}H^{n+1}_{dR}(V)$, then we will obtain the desired basis by applying the residue map to the generators and reduce the set of generators to a basis. The residue map can be defined as follows: by Thom-Leray isomorphism (see [Mov20, §4.6]), we have

$$H_{k-1}(M,\mathbb{Z}) \cong H_{k+1}(\mathbb{P}^{n+1}, \mathbb{P}^{n+1}\backslash M, \mathbb{Z}),$$

Writing the long exact sequence of the pair $(\mathbb{P}^{n+1}, \mathbb{P}^{n+1}\backslash M)$ and using the previous isomorphism we obtain

$$\cdots \to H_{k+1}(\mathbb{P}^{n+1},\mathbb{Z}) \to H_{k-1}(M,\mathbb{Z}) \xrightarrow{\sigma} H_k(V,\mathbb{Z}) \to H_k(\mathbb{P}^{n+1},\mathbb{Z}) \to \dots$$

Let us write this in de Rham cohomology:

$$\cdots \to H^{k+1}_{dR}(\mathbb{P}^{n+1}) \to H^{k+1}_{dR}(V) \xrightarrow{\sigma^*} H^k_{dR}(M) \to H^{k+2}_{dR}(\mathbb{P}^{n+1}) \to \dots$$

We have that

$$H^{k+1}_{dR}(V) \xrightarrow{\sigma^*} H^k_{dR}(M)_0$$

is an isomorphism. The de Rham cohomology of M is isomorphic to the de Rham cohomology of $\mathbb{P}^{n+1}$ except for the n-th cohomology group, so the only nontrivial isomorphism is

$$H^{n+1}_{dR}(V) \xrightarrow{\sigma^*} H^n_{dR}(M)_0.$$

That is why we are only interested in the middle cohomology.

Definition 1.1. The map $res = res_M := \sigma^*$ is called the **residue map**. It is uniquely characterized by

$$\int_\delta res(\omega) = \int_{\sigma(\delta)} \omega, \quad \omega \in H^{n+1}_{dR}(V), \;\; \delta \in H_n(M,\mathbb{Z}).$$

It is possible to introduce algebraically the residue map as well as the Hodge filtration $F^{k+1}H^{n+1}_{dR}(V)$ such that the Hodge filtration of V is compatible with Hodge filtration the M via the algebraic residue map.

The following theorem gives a set of generators of $H^{n+1}_{dR}(V)$.

Theorem 1.1 (Griffiths [Gri69])**.** For every $k = 0, \dots, n$ there is a natural map

$$H^0\left(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)\right) \longrightarrow H^{n+1}_{dR}(V)$$

such that the image is equal to $F^{n+1-k}H^{n+1}_{dR}(V)$. Here $\Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)$ denotes the sheaf of algebraic $(n+1)$-forms in $\mathbb{P}^{n+1}$ with pole of order at most $k+1$ along M. Consequently, every piece of the Hodge filtration $F^{n-k}H^n_{dR}(M)_0$ is generated by the residues of global forms with pole of order at most $k+1$ along M.

Remark 1.1.

- The sheaf $\Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)$ of algebraic $(n+1)$-forms in $\mathbb{P}^{n+1}$ with pole of order at most $k+1$ along M is given locally by:

$$\Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)(U) := \left\{ \frac{\omega}{F^{k+1}} \middle| \, \omega \;\; (n+1)\text{-holomorphic form on } U \text{ of degree } (k+1)deg(F) \right\},$$

with $M=\{F=0\}$.

- Consider $\Omega = \sum_{j=0}^{n+1}(-1)^j x_j \widehat{dx_j}$ with $\widehat{dx_j} := dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1}$ and $M=\{F=0\}$, then we have

$$\omega \in H^0\left(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)\right) \Longleftrightarrow w = \frac{P\Omega}{F^{k+1}},$$

where P is a homogeneous polynomials of degree $d(k+1)-(n+2)$.

- By the previous observation, we have

$$H^0\left(\mathbb{P}^{n+1}, \Omega^{n+1}_{\mathbb{P}^{n+1}}\left((k+1)M\right)\right) \cong H^0\left(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}\left(d(k+1)-n-2\right)\right),$$

where $\mathscr{O}_{\mathbb{P}^{n+1}}(j)$ denotes the Twisted sheaf of degree j. Locally the Twisted sheaf is given by

$$\mathscr{O}_{\mathbb{P}^{n+1}}(j)(U) := \left\{ \frac{P}{G^k} \middle| \, P \text{ homogeneous polynomial and } deg(P) = j + kdeg(G) \right\},$$

with $U = \mathbb{P}^{n+1} \backslash \{G=0\}$ and G homogeneous polynomial.

Theorem 1.2 (Griffiths [Gri69])**.** For every $k=0,\dots,n$ the kernel of the map

$$\begin{array}{ccc} H^0\left(\mathbb{P}^{n+1}, \mathscr{O}_{\mathbb{P}^{n+1}}\left(d(k+1)-n-2\right)\right) & \twoheadrightarrow & F^{n-k}H^n_{dR}(M)_0/F^{n+1-k}H^n_{dR}(M)_0 \\ P & \longmapsto & res\left(\frac{P\Omega}{F^{k+1}}\right) \end{array}$$

is the degree $N = d(k+1)-(n+2)$ part of the Jacobian ideal of F, $J_F^N \subset \mathbb{C}[x_0,\dots,x_{n+1}]_N$ where

$$J_F := \left\langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}} \right\rangle \subset \mathbb{C}[x_0,\dots,x_{n+1}]$$

Remark 1.2. The previous theorem implies that to choose a basis for

$$\frac{F^{n-k}H^n_{dR}(M)_0}{F^{n+1-k}H^n_{dR}(M)_0} \cong H^{n-k,k}(M)_0$$

it is enough to take the elements of the form $res\left(\frac{P\Omega}{F^{k+1}}\right)$ for $P \in \mathbb{C}[x_0,\dots,x_{n+1}]_N$ forming a basis of

$$V_F^N := \left(\frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{J_F}\right)_N.$$

In particular

$$h^{n-k,k}(M)_0 = \dim_{\mathbb{C}} V_F^N.$$

1.2 Weighted case

Let $v = (v_1, \dots, v_{n+1})$ with $v_j \in \mathbb{N}$, we define the action $G^v = \mathbb{C}^*$ on $\mathbb{C}^{n+1} \backslash \{0\}$ by

$$\lambda \cdot (x_1, \dots, x_{n+1}) := (\lambda^{v_1} x_1, \dots, \lambda^{v_{n+1}} x_{n+1}).$$

We call $v_1, \dots, v_{n+1}$ the weights.

Definition 1.2. Define the **v-weighted projective space** as the quotient

$$\mathbb{P}^v := \left(\mathbb{C}^{n+1} \backslash \{0\}\right) / G^v = \left(\mathbb{C}^{n+1} \backslash \{0\}\right) / \mathbb{C}^*.$$

We write points in $\mathbb{P}^v$ as $|x_1, \dots, x_{n+1}|_v$.

We can give another interpretation of $\mathbb{P}^v$ as follows: let $G_{v_j} := \left\{ e^{\frac{2\pi\sqrt{-1}m}{v_j}} \;\middle|\; m \in \mathbb{Z} \right\}$. The group $\bigoplus_{j=1}^{n+1} G_{v_j}$ acts discretely on the usual projective space as follows

$$(\epsilon_1, \dots, \epsilon_{n+1}) \cdot [x_1, \dots, x_{n+1}] := [\epsilon_1 x_1, \dots, \epsilon_{n+1} x_{n+1}]. \tag{1.1}$$

The quotient space $\mathbb{P}^n / \bigoplus_{j=1}^{n+1} G_{v_j}$ is canonically isomorphic to $\mathbb{P}^v$. This canonical isomorphism is given by

$$\begin{array}{ccc} \mathbb{P}^n / \bigoplus_{j=1}^{n+1} G_{v_j} & \longrightarrow & \mathbb{P}^v \\ [x_1, \dots, x_{n+1}] & \longmapsto & \left| x_1^{v_1}, \dots, x_{n+1}^{v_{n+1}} \right|_v \end{array}.$$

Let $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$ where weight$(x_j) = v_j$. We say that f is v**-weighted homogeneous** (or quasi-homogeneous) of degree d if

$$f = \sum a_\beta x^\beta$$

where $\beta = (\beta_1, \dots, \beta_{n+1})$ and $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_{n+1}^{\beta_{n+1}}$ such that $\sum_{j=1}^{n+1} v_j \beta_j = d$. Equivalently

$$f(\lambda^{v_1} x_1, \dots, \lambda^{v_{n+1}} x_{n+1}) = \lambda^d f(x_1, \dots, x_{n+1}) \quad \forall \lambda \in \mathbb{C}.$$

For any polynomial f in the weighted ring $\mathbb{C}[x_1, \dots, x_{n+1}]$, it can be written in a unique way $f = \sum_{i=0}^N f_i$ with f_i is a v-weighted homogeneous polynomial of degree i and $f_N \neq 0$, the number N is called the degree of f.

Definition 1.3. A polynomial $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is called a **tame polynomial** if there exist natural numbers $v \in \mathbb{N}^{n+1}$ such that $f = \sum_{i=0}^N f_i$ with f_i is a v-weighted homogeneous polynomial of degree i and $f_N \neq 0$ has an isolated singularity at $0 \in \mathbb{C}^{n+1}$, this means that the Milnor module

$$V_{f_N} := \frac{\mathbb{C}[x_1, \dots, x_{n+1}]}{\left\langle \frac{\partial f_N}{\partial x_i} \right\rangle_{i=1,\dots,n+1}}$$

is finitely generated.

We can quasi-homogenize f, namely

$$F(x_0,\dots,x_{n+1}) = x_0^N f\left(\frac{x_1}{x_0^{v_1}},\dots,\frac{x_{n+1}}{x_0^{v_{n+1}}}\right).$$

Thus, F is $(1,v)$-weighted homogeneous. On the other hand, let $\mathbb{P}^{(1,v)}$ be the projective space of weight $(1,v)=(1,v_1,\dots,v_{n+1})$, $\mathbb{P}^{(1,v)}$ is a compactification of $\mathbb{C}^{n+1}=\{(x_1,\dots,x_{n+1})\}$ with coordinates

$$\begin{array}{ccc} \mathbb{P}^{(1,v)}\backslash\{x_0=0\} & \longmapsto & \mathbb{C}^{n+1} \\ [x_0,\dots,x_{n+1}] & \longmapsto & \left(\frac{x_1}{x_0^{v_1}},\dots,\frac{x_{n+1}}{x_0^{v_{n+1}}}\right). \end{array}$$

Therefore we can regard $\{f=0\}$ as an affine open subset in $\{F=0\}\subset\mathbb{P}^{(1,v)}$.

Definition 1.4. We say that a subset $W\subseteq\mathbb{P}^v$ is a **weighted projective variety** if

$$W=\left\{p\in\mathbb{P}^v\,\middle|\, f_j(p)=0\ \ j=1,\dots,N \text{ and } f_j \text{ is } v\text{-weighted homogeneous}\right\}.$$

Let g be a v-weighted homogeneous polynomial of degree d. Then $D=\{g=0\}\subset\mathbb{P}^v$ is a weighted hypersurface. Suppose that g has an isolated singularity at $0\in\mathbb{C}^{n+1}$. Let $\widehat{g}=g(x_1^{v_1},\dots,x_{n+1}^{v_{n+1}})$, this is a homogeneous polynomial of degree d, $\widehat{g}$ has an isolated singularity at $0\in\mathbb{C}^{n+1}$ and so $\widehat{D}=\{\widehat{g}=0\}\subseteq\mathbb{P}^n$ is smooth variety. It turns out that $\widehat{D}$ is invariant under the action (1.1) and

$$D=\widehat{D}\Big/\bigoplus_{j=1}^{n+1}G_{v_j}\subseteq\mathbb{P}^v$$

Definition 1.5. The **de Rham cohomology of** $D\subseteq\mathbb{P}^v$ is the invariant part of the de Rham cohomology of $\widehat{D}$ under the action of the discrete group $\bigoplus_{j=1}^{n+1}G_{v_j}$ on $\widehat{D}$ given in (1.1), that is

$$H^m_{dR}(D):=\left\{\omega\in H^m_{dR}(\widehat{D})\,\middle|\,\epsilon^*\omega=\omega\ \ \forall\epsilon\in\bigoplus_{j=1}^{n+1}G_{v_j}\right\}.$$

A similar definition is made for the de Rham cohomology $H^m_{dR}(\mathbb{P}^v\backslash D)$.

Inspired by the smooth case, we consider $H^{n-1}_{dR}(D)_0:=Im(H^n_{dR}(\mathbb{P}^v\backslash D)\xrightarrow{res}H^{n-1}_{dR}(D))$ which has a filtration induced by the residue map. The following theorem gives a set of generators of $H^n_{dR}(D)_0$.

Theorem 1.3 (Griffiths [Gri69], Steenbrink [Ste77])**.** Let $g(x_1,\dots,x_{n+1})$ be a quasi-homogeneous polynomial of degree d, weight $v=(v_1,\dots,v_{n+1})$ with an isolated singularity at $0\in\mathbb{C}^{n+1}$ and D defined by g. For every $k=0,\dots,n-1$ there is a natural map

$$H^0\left(\mathbb{P}^v,\Omega^n_{\mathbb{P}^v}((k+1)D)\right)\longrightarrow H^n_{dR}(\mathbb{P}^v\backslash D)$$

such that the image is equal to $F^{n-k}H^n_{dR}(\mathbb{P}^v\backslash D)$. Consequently, every piece of the Hodge filtration $F^{n-1-k}H^{n-1}_{dR}(D)_0$ is generated by the residues of global forms with pole of order at most $k+1$ along D, this is generated by

$$res\left(\frac{P\Omega}{g^j}\right);\;\; j \le k+1$$

where P is a v-weighted homogeneous polynomial of degree $dj - \sum_{i=1}^{n+1} v_i$ and $\Omega = \sum_{j=1}^{n+1}(-1)^{j-1}v_j x_j \widehat{dx_j}$ with $\widehat{dx_j} := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1}$.

Corollary 1.1. Let $f(x_1, \ldots, x_{n+1})$ be a tame polynomial of degree d and D be the hypersurface defined by the quasi-homogenization F of f. Let $U := \{f = 0\}$ be the affine part of D. The $Im(H^n_{dR}(D)_0 \longrightarrow H^n_{dR}(U))$ has a filtration F_0^{n+1-k} generated by

$$\bigcup_{j=1}^{k}\left\{res\left(\frac{w_\beta}{f^j}\right)\Big| A_\beta < j\right\},$$

where $\omega_\beta = x^\beta dx := x_1^{\beta_1}\ldots x_{n+1}^{\beta_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1}$, $1 \le k \le n+1$ and $A_\beta := \sum_{j=1}^{n+1}(\beta_j + 1)\frac{v_j}{d}$.

Proof. Consider $D_j := \left\{x_0^{\beta_0}x^\beta \Big| \sum_{i=0}^{n+1}\beta_i v_i = dj - \sum_{i=0}^{n+1} v_i\right\}$. So applying previous theorem to $F(x_0, \ldots, x_{n+1})$ we obtain that $F^{n+1-k}H^n_{dR}(D)_0$ is generated by

$$\bigcup_{j=1}^{k}\left\{res\left(\frac{x_0^{\beta_0}x^\beta\Omega}{F^j}\right)\Big| x_0^{\beta_0}x^\beta \in D_j\right\}.$$

Now, observe that

$$\left\{\frac{x_0^{\beta_0}x^\beta\Omega}{F^j}\Bigg|_{x_0=1}\right\}_{x_0^{\beta_0}x^\beta \in D_j} = \left\{\frac{\omega_\beta}{f^j}\right\}_{A_\beta<j}.$$

■

Theorem 1.4 (Griffiths [Gri69], Steenbrink [Ste77]). Let $g(x_1, \ldots, x_{n+1})$ be a weighted homogeneous polynomial of degree d, weight $v = (v_1, \ldots, v_{n+1})$ with an isolated singularity at $0 \in \mathbb{C}^{n+1}$ and D defined by g. We have

$$H^n_{dR}(\mathbb{P}^v \backslash D) \cong \frac{H^0\left(\mathbb{P}^v, \Omega^n(*D)\right)}{dH^0\left(\mathbb{P}^v, \Omega^{n-1}(*D)\right)}$$

and under the above isomorphism

$$\frac{F^{n-k+1}}{F^{n-k+2}} \cong \frac{H^0\left(\mathbb{P}^v, \Omega^n(kD)\right)}{dH^0\left(\mathbb{P}^v, \Omega^{n-1}((k-1)D)\right) + H^0\left(\mathbb{P}^v, \Omega^n((k-1)D)\right)},$$

where $0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H_{dR}(\mathbb{P}^v \backslash D)$ is the Hodge filtration of $H_{dR}(\mathbb{P}^v \backslash D)$. Let $\left\{x^\beta \Big| \beta \in I\right\}$ be a basis of monomials for the Milnor module

$$V_g := \frac{\mathbb{C}[x_1, \ldots, x_{n+1}]}{\left\langle \frac{\partial g}{\partial x_i}\right\rangle_{i=1,\ldots,n+1}}.$$

A basis of $\frac{F^{n-k+1}}{F^{n-k+2}}$ is given by

$$\left\{\frac{x^\beta \Omega}{g^k}\right\}_{\substack{A_\beta = k\\ \beta\in I}},$$

where $\Omega = \sum_{j=1}^{n+1}(-1)^j v_j x_j \widehat{dx_j}$ with $\widehat{dx_j} := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1}$ and $A_\beta = \sum_{j=1}^{n+1}(\beta_j+1)\frac{v_j}{d}$.

Example 1.1. Let $g = x_0^{m_0} + \cdots + x_{n+1}^{m_{n+1}}$ be a weighted homogeneous polynomial of degree d and weight $v = (v_0, \dots, v_{n+1})$. By the previous theorem $H^n(D)_0$ has a filtration $0 = F^{n+1} \subset F^n \subset \cdots \subset F^0 = H^n(D)_0$. A basis of F^{n+1-k} is given by

$$\bigcup_{j=1}^{k}\left\{res\left(\frac{x_0^{\beta_0}x^\beta\Omega}{g^j}\right)\right\}_{\substack{A_{(\beta_0,\beta)}=j\\ (\beta_0,\beta)\in I_{m_0}\times I}}$$

where $I = I_{m_1} \times \cdots \times I_{m_{n+1}}$ and $I_{m_j} = \{0, 1, \dots, m_j - 2\}$. The restriction $x_0 = 1$ induces a filtration F_0^{n+1-k} in $Im(H^n_{dR}(D)_0 \longrightarrow H^n_{dR}(U))$ with basis given by

$$\bigcup_{j=1}^{k}\left\{res\left(\frac{\omega_\beta}{(g^j)|_{x_0=1}}\right)\right\}_{\substack{j-1<A_\beta<j\\ \beta\in I}}.$$

1.3 Hodge cycles

Let M be a smooth projective hypersurface and Y be a smooth hyperplane section of M. Writing the long exact sequence of the pair (M, V), where $V = M\backslash Y$, and using the Thom-Leray isomorphism we have

$$\cdots \to H_{n-1}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_n(V,\mathbb{Z}) \xrightarrow{i} H_n(M,\mathbb{Z}) \xrightarrow{\tau} H_{n-2}(Y,\mathbb{Z}) \to \dots,$$

where the map τ is the intersection with Y.

Definition 1.6. An element $\delta \in H_n(V,\mathbb{Q})$ is called a **cycle at infinity** if $\delta \in Ker(H_n(V,\mathbb{Q}) \xrightarrow{i} H_n(M,\mathbb{Q}))$. We denote

$$\begin{aligned} H_n(V,\mathbb{Q})_\infty &:= Ker(H_n(V,\mathbb{Q}) \xrightarrow{i} H_n(M,\mathbb{Q})) \\ &= Im(H_{n-1}(Y,\mathbb{Z}) \xrightarrow{\sigma} H_n(V,\mathbb{Q})). \end{aligned}$$

We denote the primitive homology (dual to the primitive cohomology, see [Mov20, §5.7]) by

$$\begin{aligned} H_n(M,\mathbb{Q})_0 &:= \{x \in H_n(M,\mathbb{Q}) |\ [Y]\cdot x = 0\} \\ &= Ker(H_n(M,\mathbb{Q}) \xrightarrow{\tau} H_{n-2}(Y,\mathbb{Q})). \end{aligned}$$

Thus, we can consider the primitive homology as

$$\begin{aligned} H_n(M,\mathbb{Q})_0 :&= Ker(H_n(M,\mathbb{Q}) \xrightarrow{\tau} H_{n-2}(Y,\mathbb{Q})) \\ &= Im(H_n(V,\mathbb{Q}) \xrightarrow{i} H_n(M,\mathbb{Q})) \\ &\cong \frac{H_n(V,\mathbb{Q})}{Ker(H_n(V,\mathbb{Q}) \xrightarrow{i} H_n(M,\mathbb{Q}))}. \end{aligned}$$

In conclusion the study of the primitive homology of M can be done by studying the homology of its affine part, since

$$H_n(M,\mathbb{Q})_0 \cong \frac{H_n(V,\mathbb{Q})}{H_n(V,\mathbb{Q})_\infty}. \tag{1.2}$$

On the other hand, we have the Hodge filtration: $0 = F^{n+1} \subset F^n \subset \cdots \subset F^1 \subset F^0 = H^n_{dR}(M)$ with $F^k = F^k H^n_{dR}(M) := H^{n,0} + H^{n-1,1} + \cdots + H^{k,n-k}$ where $H^{k,n-k} := H^{k,n-k}(M)$, which allows us to define Hodge cycles.

Definition 1.7. A cycle $\delta \in H_n(M,\mathbb{Q})$ is called a **Hodge cycle** if

$$\int_\delta F^{\frac{n}{2}+1} = 0.$$

We denote by $\text{Hod}_n(M,\mathbb{Q})$ the space of Hodge cycles in $H_n(M,\mathbb{Q})$.

Now, using [Mov20, Proposition 5.10] and equation (1.2) we have

$$\text{Hod}_n(M,\mathbb{Q})_0 := \text{Hod}_n(M,\mathbb{Q}) \cap H_n(M,\mathbb{Q})_0 \cong \frac{\left\{\delta \in H_n(V,\mathbb{Q}) | \int_\delta F_0^{\frac{n}{2}+1} = 0\right\}}{\left\{\delta \in H_n(V,\mathbb{Q}) | \int_\delta F_0^0 = 0\right\}}, \tag{1.3}$$

with $F_0^k = F^k \cap H^n_{dR}(M)_0$.

Let us go back to our case of interest: remember that $\omega_\beta = x^\beta dx := x_1^{\beta_1} \ldots x_{n+1}^{\beta_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1}$ and $A_\beta = \sum_{j=1}^{n+1} (\beta_j + 1)\frac{v_j}{d}$. The polynomial $f = g(x) + P(y)$ is a tame polynomial in $\mathbb{C}[x_1, \ldots, x_{n+1}]$, with $weight(x_j) = v_j := \frac{d}{m_j}$ and $d = lcm(m_1, \cdots, m_{n+1})$ where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$, and $P(y) : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $m = m_{n+1}$. Inspired by Corollary 1.1 and equation (1.3), we have the next definition.

Definition 1.8. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$. We define the Hodge cycle space as

$$\text{Hod}_n(X,\mathbb{Q})_0 := \frac{\left\{\delta \in H_n(U,\mathbb{Q}) | \int_\delta res\left(\frac{\omega_\beta}{f^j}\right) = 0, A_\beta < j, 1 \leq j \leq \frac{n}{2}\right\}}{\left\{\delta \in H_n(U,\mathbb{Q}) | \int_\delta res\left(\frac{\omega_\beta}{f^j}\right) = 0, A_\beta < j, 1 \leq j \leq n+1\right\}},$$

with $U := \{f = 0\} \subset D := \{F = 0\} \subset \mathbb{P}^{(1,v)}$.

If $X = M$, it follows by Theorem 1.4 that this definition coincides with the classical definition of Hodge cycles (see equation (1.3)). For instance, if X is defined by $f = x_1^d + \cdots + x_n^d + x_{n+1}^d + 1$.

CHAPTER 2

Integration over joint cycles

The object of this chapter is to introduce a techniques to calculate periods. In §2.1 we will describe a particular basis of homology, namely joint cycles. We will give a formula for the integral on one of these cycles, which will be used later in the calculation of periods on perturbations of Fermat variety. In §2.2 we will express the cycles on an affine curve in terms of joint cycles. Finally, in §2.3 we will give a basis for the homology and cohomology of affine Fermat varieties and we will compute the integral on these bases, which will naturally arise in the calculation of periods on perturbations of Fermat varieties. The two main results of this chapter that we will use frequently in later chapters are Theorem 2.1 and Proposition 2.6. The main references for this chapter are [Mov20, AVGZ88].

2.1 Joint cycles

Let $h:\mathbb{C}^n\to\mathbb{C}$ be a holomorphic function with non-degenerate critical points and all critical values are different, this means h is a Morse function. By Morse lemma, there is a local coordinate system around every non-degenerate critical point p such that $h(z)=h(p)+z_1^2+\cdots+z_n^2$.

Let $\gamma:[0,1]\to\mathbb{C}$ be a simple path such that $\gamma(1)$ is a critical value of h and $\gamma(t)$ is a regular value for $t\in[0,1)$. For the parameter t near 1, we fix the sphere $S(t)=\sqrt{\gamma(t)-\gamma(1)}S^{n-1}$ in the level set $\{h(z)=\gamma(t)\}$ where

$$S^{n-1}=\left\{(z_1,\dots,z_n)\middle|\ \sum_{j=1}^n z_j^2=1,\ \ Imz_j=0\right\}$$

is the $(n-1)$-dimensional sphere. This defines a family of $(n-1)$-dimensional spheres $S(t)\subset\{h=\gamma(t)\}$ for all $t\in[0,1)$. Note that for $t=1$ the sphere $S(t)$ reduces to the critical point p.

Definition 2.1. The cycle $\delta\in H_{n-1}(\{h=\gamma(0)\})$ induced by the $(n-1)$- dimensional sphere $S(0)$ is called **vanishing cycle** along the path γ.

Since the set of Morse functions is dense in the set of holomorphic functions, we can define vanishing cycles for holomorphic functions with degenerate critical points. For more details see [AVGZ88, Chapter 1,2]. Vanishing cycles satisfy the following fact: let $h:\mathbb{C}^n\to\mathbb{C}$ be a tame polynomial with $b \in \mathbb{C}$ regular value. The $(n-1)$-homology group of the fiber over b, $H_{n-1}(h^{-1}(b),\mathbb{Z})$, is freely generated by the vanishing cycles (see [AVGZ88, Chapter 1,2], [Mov20, §7.5]).

Let $h \in \mathbb{C}[x]$ and $l \in \mathbb{C}[y]$ be two polynomials with $x = (x_1, \dots, x_{n+1})$ and $y = (y_1, \dots, y_{m+1})$. Let C_1 denote the set of critical values of h (resp. C_2 denote the set of critical values of l). We assume that $C_1 \cap C_2 = \emptyset$, which implies that the variety

$$M := \{(x,y) \in \mathbb{C}^{n+1} \times \mathbb{C}^{m+1} | h(x) = l(y)\}$$

is smooth. Fix a regular value $b \in \mathbb{C} \setminus (C_1 \cup C_2)$ of h and l. Let $\delta_{1b} \in H_n(h^{-1}(b),\mathbb{Z})$ and $\delta_{2b} \in H_m(l^{-1}(b),\mathbb{Z})$ be two vanishing cycles and t_s, $s \in [0,1]$ be a path in $\mathbb{C}$ such that it starts from a point in C_1, crosses b and ends in a point of C_2 and never crosses $C_1 \cup C_2$ except at the mentioned cases. We assume that δ_{1b} vanishes along t^{-1} when s tends to 0 and δ_{2b} vanishes along t when s tend to 1.

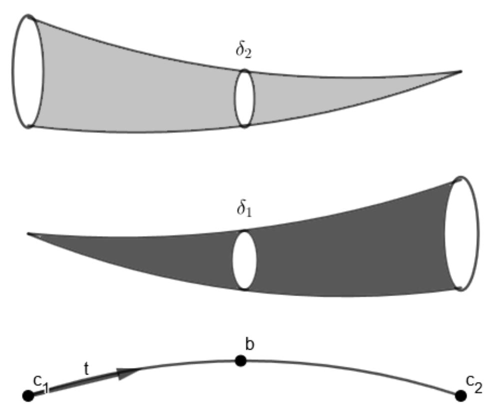

Figure 2.1: Joint of vanishing cycles

Definition 2.2. The cycle

$$\delta_1 * \delta_2 = \delta_1 *_t \delta_2 := \bigcup_{s \in [0,1]} \delta_{1t_s} \times \delta_{2t_s} \in H_{n+m+1}(M,\mathbb{Z})$$

is called the **joint cycle** of δ_{1b} and δ_{2b} along t. We call the triple $(t, \delta_1, \delta_2) = (t, \delta_{1t}, \delta_{2t})$ an admissible triple.

We take a system of distinguished paths of λ_c, where λ_c start at b and ends at $c \in C_1 \cup C_2$ (see Figure 2.2). This means

1. Each path λ_c has no self intersection points.

2. Two distinct paths λ_{c_i} and λ_{c_j} meet only at their common origin $\lambda_{c_i}(0) = b = \lambda_{c_j}(0)$.

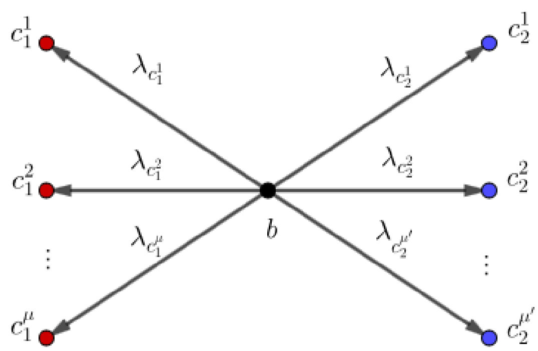

Figure 2.2: A system of distinguished paths

Let

$$\delta_1^1, \delta_1^2, \dots, \delta_1^\mu \in H_n(h^{-1}(b), \mathbb{Z}) \text{ and } \delta_2^1, \delta_2^2, \dots, \delta_2^{\mu'} \in H_n(l^{-1}(b), \mathbb{Z})$$

be the corresponding vanishing cycles.

Theorem 2.1. Let h and l be polynomials tame with disjoint set of critical values then $H_{n+m+1}(M, \mathbb{Z})$ is freely generated by

$$\delta_1^i * \delta_2^j, \quad i = 1, \cdots, \mu, \quad j = 1, \cdots, \mu',$$

where we have taken the admissible triples

$$\left(\lambda_{c_2^j} \lambda_{c_1^i}^{-1}, \delta_1^i, \delta_2^j\right), \quad c_1^i \in C_1, \quad c_2^j \in C_2.$$

Proof. See [Mov20, Theorem 7.4]. For a local version see [AVGZ88, Theorem 2.9]. ■

Definition 2.3. Let ω be a holomorphic $(n+1)$-form and f be a holomorphic function in a region of $\mathbb{C}^{n+1}$. We define the Gelfand-Leray form $\frac{\omega}{df}$ as the form ω' that satisfies $\omega = df \wedge \omega'$.

If p is not a critical point of f, the Gelfand-Leray always exists around p. Indeed, The Gelfand-Leray around p of $\omega = g dx_1 \wedge \cdots \wedge dx_{n+1}$ is given by

$$\frac{\omega}{df} = (-1)^n \frac{g dx_1 \wedge \cdots \wedge dx_n}{f_{x_{n+1}}},$$

where g meromorphic function, and $f_{x_{n+1}}$ is nonzero around p, with $f_{x_{n+1}}$ is the partial derivative of f with respect to x_{n+1}. Also, if $\{f=0\}$ is smooth, the restriction of $\frac{\omega}{df}$ to $\{f=0\}$ does not depend on the choice of the form ω' satisfying $\omega = df \wedge \omega'$. Our main example of Gelfand Leray form will be

$$\frac{y^\alpha z^\gamma dy \wedge dz}{d(P(y)-z^q)} = \frac{y^\alpha z^{\gamma-q+1}dy}{q}.$$

The following results in this section will simplify the integrals and will allow us to calculate them. For a more extensive exposition see [Mov20, §13.8].

Proposition 2.1. Let $h \in \mathbb{C}[x]$ and $l \in \mathbb{C}[y]$ be two tame polynomials. Let ω_1 be a $(n+1)$-form in $\mathbb{C}^{n+1}$ and ω_2 be a $(m+1)$-form in $\mathbb{C}^{m+1}$. Let also $(t_s, \delta_{1b}, \delta_{2b})$ be an admisible triple and

$$I_1(t_s) := \int_{\delta_{1,t_s}} \frac{\omega_1}{dh}, \quad I_2(t_s) := \int_{\delta_{2,t_s}} \frac{\omega_2}{dl}.$$

Then

$$\int_{\delta_{1b} *_t \delta_{2b}} \frac{\omega_1 \wedge \omega_2}{d(h-l)} = \int_{t_s} I_1(t_s) I_2(t_s) dt_s.$$

Proof. First, note that

$$\begin{aligned}\omega_1 \wedge \omega_2 &= dh \wedge \frac{\omega_1}{dh} \wedge dl \wedge \frac{\omega_2}{dl} \\ &= d(h-l) \wedge \frac{\omega_1}{dh} \wedge dl \wedge \frac{\omega_2}{dl},\end{aligned}$$

this means that $\frac{\omega_1 \wedge \omega_2}{d(h-l)} = \frac{\omega_1}{dh} \wedge dl \wedge \frac{\omega_2}{dl}$. But over the joint cycle $\delta_{1b} *_t \delta_{2b}$ we have $t_s = h(x) = l(y)$, therefore

$$\begin{aligned}\int_{\delta_{1b} *_t \delta_{2b}} \frac{\omega_1 \wedge \omega_2}{d(h-l)} &= (-1)^m \int_{\delta_{1b} *_t \delta_{2b}} \frac{\omega_1}{dh} \wedge \frac{\omega_2}{dl} \wedge dt_s \\ &= (-1)^m \int_{t_s} I_1(t_s) I_2(t_s) dt_s.\end{aligned}$$

■

For a polynomial h and $\delta \in H_n(\{h=b\}, \mathbb{Z})$, consider

$$p(\beta, \delta) = p(\{h=b\}, \beta, \delta) := \int_\delta \frac{x^\beta dx}{dh}.$$

Now, we describe how to reduce a higher dimensional integral to a lower-dimensional one. This corresponds to [Mov20, Propositions 13.9 and 13.10]. The proofs given here are slightly different and in a simpler language than those given in [Mov20].

Proposition 2.2. Let $P(y) = P(y_1, \ldots, y_{m+1})$ be a tame polynomial with b regular value of P and $g(x) = g(x_1, \ldots, x_{n+1})$ be a quasi-homogeneous tame polynomial of degree d. Let $\delta_1 \in H_m(\{P(y) = b\}, \mathbb{Z})$, $\delta_2 \in H_n(\{g(x) = b\}, \mathbb{Z})$ be vanishing cycles and t_s, $s \in [0,1]$ a path in the $\mathbb{C}$-plane which connects a critical value of P to 0 (the unique critical value of g) with $b \in t_s$. We assume that δ_1 vanishes along t^{-1} and δ_2 vanishes along t. Then

$$\int_{\delta_1 *_t \delta_2} \frac{y^\alpha x^\beta dy \wedge dx}{d(P-g)} = \begin{cases} \frac{p(\{g=b\},\beta,\delta_2)}{p(\{z^q=b\},\gamma,\delta_3)} \int_{\delta_1 *_t \delta_3} \frac{y^\alpha z^\gamma dy \wedge dz}{d(P-z^q)} & A_\beta \notin \mathbb{N} \\ \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta - 1}} \int_{\widetilde{\delta}_1} \widetilde{\omega} & A_\beta \in \mathbb{N} \end{cases},$$

where $\delta_3 = b^{1/q}[\zeta_q] - [1] \in H_0(\{z^q = b\}, \mathbb{Z})$, q and γ are given by the equality $A_\beta := \sum_{i=1}^n (\beta_i + 1)\frac{v_i}{d} = \frac{\gamma+1}{q}$, and $p(\gamma, \delta_3) = b^{A_\beta - 1}\frac{\zeta_q^{\gamma+1}-1}{q}$, in the first case. In the second case, $\widetilde{\delta}_1 \in H_m(\{P = 0\}, \mathbb{Z})$ is the monodromy of δ_1 along the path t_s and $\widetilde{\omega}$ is a m-form such that $d\widetilde{\omega} = P^{A_\beta - 1} y^\alpha dy$.

Proof. Let us first see that

$$I_2(t_s) = \int_{\delta_{2,t_s}} \frac{x^\beta dx}{dg} = \left(\int_{\delta_2} \frac{x^\beta dx}{dg}\right)\left(\frac{t_s}{b}\right)^{A_\beta - 1} = p(\beta, \delta_2)\left(\frac{t_s}{b}\right)^{A_\beta - 1}. \tag{2.1}$$

For this consider the biholomorphism

$$\begin{array}{rccc} \phi_t : & \{g(x) = b\} & \longmapsto & \{g(x) = t\} \\ & (x_1, \ldots, x_{n+1}) & \longmapsto & \left(\left(\frac{t}{b}\right)^{v_1/d} x_1, \cdots, \left(\frac{t}{b}\right)^{v_{n+1}/d} x_{n+1}\right). \end{array}$$

Observe that $\delta_{2,t} = (\phi_t)_*(\delta_2)$. A straightforward calculation allows us to conclude equation (2.1). This is also true for z^q since it is a homogeneous tame polynomial. Using the above, we have

$$\begin{aligned} \int_{\delta_1 *_t \delta_2} \frac{y^\alpha x^\beta dy \wedge dx}{d(P-g)} &= \int_{\delta_1 *_t \delta_2} \frac{(y^\alpha dy) \wedge (x^\beta dx)}{d(P-g)} \\ &= \int_{t_s} I_1(t_s) I_2(t_s) dt_s \\ &= p(\{g=b\}, \beta, \delta_2) \int_{t_s} \left(\frac{t_s}{b}\right)^{A_\beta - 1} I_1(t_s) dt_s \\ &= p(\{g=b\}, \beta, \delta_2) \int_{t_s} \left(\frac{t_s}{b}\right)^{\frac{\gamma+1}{q} - 1} I_1(t_s) dt_s \\ &= \frac{p(\{g=b\}, \beta, \delta_2)}{p(\{z^q=b\}, \gamma, \delta_3)} \int_{t_s} I_1(t_s) I_3(t_s) dt_s \\ &= \frac{p(\{g=b\}, \beta, \delta_2)}{p(\{z^q=b\}, \gamma, \delta_3)} \int_{\delta_1 *_t \delta_3} \frac{y^\alpha z^\gamma dy \wedge dz}{d(P-z^q)}. \end{aligned}$$

Which allows us to conclude the first part. For the second part, note that we cannot use the same reasoning because we would divide by zero.

$$
\begin{aligned}
\int_{\delta_1 *_t \delta_2} \frac{y^\alpha x^\beta dy \wedge dx}{d(P-g)} &= p(\{g=b\},\beta,\delta_2) \int_{t_s} \left(\int_{\delta_{1,t_s}} \frac{y^\alpha dy}{dP} \right) \left(\frac{t_s}{b} \right)^{A_\beta - 1} dt_s \\
&= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}} \int_{t_s} \left(\int_{\delta_{1,t_s}} \frac{P^{A_\beta-1} y^\alpha dy}{dP} \right) dt_s \text{ Because } P(y)=t_s \\
&= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}} \int_{\Delta} P^{A_\beta-1} y^\alpha dy \text{ Because } P(y)=t_s \\
&= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}} \int_{\Delta} d\widetilde{\omega} \\
&= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}} \int_{\widetilde{\delta}_1} \widetilde{\omega} \quad \text{By Stokes' theorem,}
\end{aligned}
$$

where

$$
\Delta = \bigcup_{s\in[0,1]} \delta_{1,t_s} \in H_{m+1}(\mathbb{C}^{m+1}, P^{-1}(0), \mathbb{Z})
$$

is the Lefschetz thimble with the boundary $\widetilde{\delta}_1$. ∎

Remark 2.1. Here a fact that we use often and that the reader should know is: $res\left(\frac{\omega}{f}\right) = \frac{\omega}{df}$. For example, for $A_\beta \notin \mathbb{N}$, we can rewrite the formula in the previous proposition:

$$
\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy \wedge dx}{P-g} \right) = \frac{p(\{g=b\},\beta,\delta_2)}{p(\{z^q=b\},\gamma,\delta_3)} \int_{\delta_1 *_t \delta_3} res\left(\frac{y^\alpha z^\gamma dy \wedge dz}{P-z^q} \right),
$$

Proposition 2.3. With notations of Proposition 2.2 we have

$$
\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy \wedge dx}{(P-g)^k} \right) = \begin{cases}
\frac{p(\{g=b\},\beta,\delta_2)}{p(\{z^q=b\},\gamma,\delta_3)} \int_{\delta_1 *_t \delta_3} res\left(\frac{y^\alpha z^\gamma dy \wedge dz}{(P-z^q)^k} \right) & A_\beta \notin \mathbb{N}, \\
\frac{(-1)^{A_\beta-1} p(\{g=b\},\beta,\delta_2)(A_\beta-1)!(k-A_\beta-1)!}{b^{A_\beta-1}(k-1)!} \int_{\widetilde{\delta}_1} res\left(\frac{y^\alpha dy}{P^{k-A_\beta}} \right) & k > A_\beta \in \mathbb{N}, \\
\frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}(k-1)!} \int_{\widetilde{\delta}_1} \widetilde{\omega}_1 & k \le A_\beta \in \mathbb{N},
\end{cases}
$$

where $\widetilde{\omega}_1$ is a m-form such that $d\widetilde{\omega}_1 = (-1)^{k-1}(A_\beta - k + 1)_{k-1} P^{A_\beta - k} y^\alpha dy$.

Proof. We introduce a new parameter s and observe that

$$
\frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial s^{k-1}} \int_{\sigma(\delta_1 *_t \delta_2)} \frac{\omega}{f-s} = \int_{\sigma(\delta_1 *_t \delta_2)} \frac{\omega}{(f-s)^k},
$$

where σ is the "dual" of the residue map (see Definition 1.1). Now using this and the previous proposition, for $A_\beta \notin \mathbb{N}$ we have

$$\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{(P-g)^k}\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial s^{k-1}}\Big|_{s=0}\int_{\sigma(\delta_1 *_t \delta_2)}\frac{y^\alpha x^\beta dy\wedge dx}{P-s-g}$$
$$= \frac{p(\{g=b\},\beta,\delta_2)}{p(\{z^q=b\},\gamma,\delta_3)}\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial s^{k-1}}\Big|_{s=0}\int_{\sigma(\delta_1 *_t \delta_3)}\frac{y^\alpha z^\gamma dy\wedge dz}{P-s-z^q}$$
$$= \frac{p(\{g=b\},\beta,\delta_2)}{p(\{z^q=b\},\gamma,\delta_3)}\int_{\delta_1 *_t \delta_3} res\left(\frac{y^\alpha z^\gamma dy\wedge dz}{(P-z^q)^k}\right).$$

This proves the first part. Before continuing, note that

$$\frac{\partial}{\partial s}\int_{\widetilde{\delta}_{1s}}\widetilde{\omega}(s) = \int_{\widetilde{\delta}_{1s}}\left(\frac{\partial}{\partial s}\widetilde{\omega} + res\left(\frac{d\widetilde{w}}{P-s}\right)\right),$$

where $\widetilde{\delta}_{1s}\in H_m(\{P-s=0\},\mathbb{Z})$ and $\widetilde{\omega}$ is the m-form such that $d\widetilde{\omega}=(P-s)^{A_\beta-1}y^\alpha dy$. Indeed

$$\frac{\partial}{\partial s}\int_{\widetilde{\delta}_{1s}}\widetilde{\omega}(s) = \frac{\partial}{\partial s}\int_{\sigma(\widetilde{\delta}_{1s})} dP\wedge\frac{\widetilde{\omega}}{P-s}$$
$$= \int_{\sigma(\widetilde{\delta}_{1s})} dP\wedge\frac{\frac{\partial}{\partial s}\widetilde{\omega}}{P-s} + \frac{dP\wedge\widetilde{\omega}}{(P-s)^2}$$
$$= \int_{\widetilde{\delta}_{1s}}\left(\frac{\partial}{\partial s}\widetilde{\omega} + res\left(\frac{d\widetilde{w}}{P-s}\right)\right).$$

In addition, in this case the second term on the right side is zero. For the $k>A_\beta$, successively applying the above we have

$$\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{(P-g)^k}\right) = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial s^{k-1}}\Big|_{s=0}\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{P-s-g}\right)$$
$$= \frac{p(\{g=b\},\beta,\delta_2)}{(k-1)!b^{A_\beta-1}}\frac{\partial^{k-1}}{\partial s^{k-1}}\Big|_{s=0}\int_{\widetilde{\delta}_{1s}}\widetilde{\omega}(s)\ \ \text{By Proposition 2.2.}$$
$$= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}(k-1)!}\frac{\partial^{k-A_\beta-1}}{\partial s^{k-A_\beta-1}}\Big|_{s=0}\int_{\widetilde{\delta}_{1s}}\left(\frac{\partial^{A_\beta}}{\partial s^{A_\beta}}\widetilde{\omega} + res\left(\frac{d\left(\frac{\partial^{A_\beta-1}}{\partial s^{A_\beta-1}}\widetilde{\omega}\right)}{P-s}\right)\right).$$

Using that

$$\frac{\partial^j}{\partial s^j}d\widetilde{\omega} = (-1)^j(A_\beta-1)(A_\beta-2)\dots(A_\beta-j)(P-s)^{A_\beta-j-1}y^\alpha dy,$$

then

$$\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{(P-g)^k}\right) = \frac{(-1)^{A_\beta-1}p(\{g=b\},\beta,\delta_2)(A_\beta-1)!}{b^{A_\beta-1}(k-1)!}\frac{\partial^{k-A_\beta-1}}{\partial s^{k-A_\beta-1}}\Big|_{s=0}\int_{\widetilde{\delta}_{1s}} res\left(\frac{y^\alpha dy}{P-s}\right)$$
$$= \frac{(-1)^{A_\beta-1}p(\{g=b\},\beta,\delta_2)(A_\beta-1)!(k-A_\beta-1)!}{b^{A_\beta-1}(k-1)!}\int_{\widetilde{\delta}_1} res\left(\frac{y^\alpha dy}{P^{k-A_\beta}}\right).$$

For the last case, we have

$$\begin{aligned}\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{(P-g)^k}\right) &= \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial s^{k-1}}\bigg|_{s=0}\int_{\delta_1 *_t \delta_2} res\left(\frac{y^\alpha x^\beta dy\wedge dx}{P-s-g}\right)\\ &= \frac{p(\{g=b\},\beta,\delta_2)}{(k-1)!b^{A_\beta -1}}\frac{\partial^{k-1}}{\partial s^{k-1}}\bigg|_{s=0}\int_{\widetilde{\delta}_{1s}}\widetilde{\omega}(s) \text{ By Proposition 2.2.}\\ &= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}(k-1)!}\int_{\widetilde{\delta}_1}\left(\frac{\partial^{k-1}}{\partial s^{k-1}}\bigg|_{s=0}\widetilde{\omega}+res\left(\frac{d\left(\frac{\partial^{k-2}}{\partial s^{k-2}}\widetilde{\omega}\right)}{P-s}\right)\bigg|_{s=0}\right)\\ &= \frac{p(\{g=b\},\beta,\delta_2)}{b^{A_\beta-1}(k-1)!}\int_{\widetilde{\delta}_1}\frac{\partial^{k-1}}{\partial s^{k-1}}\bigg|_{s=0}\widetilde{\omega}.\end{aligned}$$

Observe that $\widetilde{\omega}_1 = \frac{\partial^{k-1}}{\partial s^{k-1}}\Big|_{s=0}\widetilde{\omega}$ satisfies the property of the statement. ■

2.2 Cycles on affine curves

Let us consider $M=\{(y,z)\in\mathbb{C}^2 | z^N = P(y)\}$ with $P(y)$ polynomial such that 0 is not a critical value of $P(y)$. We know that $H_1(M,\mathbb{Z})$ is freely generated by joint cycles. More explicitly, we take a system of distinguished paths of λ_c, where λ_c start at b and ends at $c\in C\cup\{0\}$, with b non zero regular value of $P(y)$ and C the set of critical values of $P(y)$, see Figure 2.3. Let

$$\delta_1^1,\delta_1^2,\dots,\delta_1^\mu\in H_0(\{P(y)=b\},\mathbb{Z}),\ \ \delta_2^1,\delta_2^2,\dots,\delta_2^{N-1}\in H_0(\{z^N=b\},\mathbb{Z})$$

be the corresponding vanishing cycles. Thus $H_1(M,\mathbb{Z})$ is freely generated by

$$\delta_1^i * \delta_2^j,\ \ i=1,\dots,\mu,\ \ j=1,\dots,N-1,$$

where we have taken the admissible triples $(\lambda_0\lambda_c^{-1},\delta_1^i,\delta_2^j),\ \ c\in C$. In this case

$$\delta_{1\lambda_c}^i=[y_{1\lambda_c}^i - y_{2\lambda_c}^i],\ \ \delta_{2\lambda_0}^j=[z_{1\lambda_0}^j - z_{2\lambda_0}^j]$$

and

$$\delta_1^i *_{\lambda_0\lambda_c^{-1}} \delta_2^j = \bigcup_{s\in[0,1]}[y_{1\lambda_c(s)}^i - y_{2\lambda_c(s)}^i]\times[z_{1\lambda_0(s)}^j - z_{2\lambda_0(s)}^j]\in H_1(M,\mathbb{Z}).$$

The projection of the cycle $\delta_1^i *_{\lambda_0\lambda_c^{-1}} \delta_2^j$ in the y-coordinate induces a path $\sigma:[0,1]\to\mathbb{C}$ that connects two roots of $P(y)$ and passes through the critical point c.

2.3 Multiple Integrals for Fermat varieties

Let $m_1,m_2,\dots,m_{n+1}$ be integers bigger than one and consider the n-th affine Fermat variety

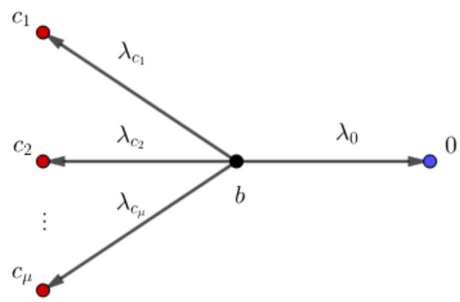

Figure 2.3: A system of distinguished paths II

$$L_b := \left\{x \in \mathbb{C}^{n+1} \middle| g(x) = b\right\} \subset \mathbb{C}^{n+1},$$

where $g = x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}}$, and b nonzero. We denote $L := L_1$. Let

$$\Delta^n := \left\{ (t_1, \cdots, t_{n+1}) \in \mathbb{R}^{n+1} \middle| \, t_j \geq 0, \, \sum_{j=1}^{n+1} t_j = 1 \right\}$$

be the standard n-simplex and let $\zeta_{m_j} = e^{\frac{2\pi\sqrt{-1}}{m_j}}$ be an m_j-th primitive root of unity. For $\alpha \in I = I_{m_1} \times I_{m_2} \times \cdots \times I_{m_{n+1}}$ with $I_m = \{0, \ldots, m-2\}$ and $a \in \{0,1\}^{n+1}$, consider

$$\begin{array}{cccc} \Delta_{\alpha+a}: & \Delta^n & \longmapsto & L \\ & (t_1, \ldots, t_{n+1}) & \longmapsto & \left(t_1^{\frac{1}{m_1}} \zeta_{m_1}^{\alpha_1 + a_1}, \ldots, t_{n+1}^{\frac{1}{m_{n+1}}} \zeta_{m_{n+1}}^{\alpha_{n+1}+a_{n+1}} \right). \end{array}$$

The formal sum

$$\delta_\alpha := \sum_a (-1)^{\sum_{i=1}^{n+1}(1-a_i)} \Delta_{\alpha+a}$$

induces a non-zero element in $H_n(L, \mathbb{Z})$. In fact

Proposition 2.4. The cycles $\{\delta_\alpha\}_{\alpha \in I}$ are a basis for the $\mathbb{Z}$-module $H_n(L, \mathbb{Z})$.

Proof. See [Mov20, Remark 7.1]. The version in the language of singularities can be found in [AVGZ88, §2.9]. The first to describe this base was Pham in [Pha65]. ■

Consider the biholomorphism $\phi_b : L \to L_b$, $\phi(x_1, \cdots, x_{n+1}) = (b^{1/m_1} x_1, \cdots, b^{1/m_{n+1}} x_{n+1})$ with b^{1/m_j} is a fixed m_j-th root of b. Let us consider $\delta_\alpha^b = (\phi_b)_*(\delta_\alpha)$. Thus

Corollary 2.1. The cycles $\{\delta_\alpha^b\}_{\alpha \in I}$ are a basis for the $\mathbb{Z}$-module $H_n(L_b, \mathbb{Z})$.

Now consider

$$\eta_\alpha := x^\alpha \sum_{j=1}^{n+1} \frac{(-1)^{j-1} x_j}{m_j} \widehat{dx_j} \tag{2.2}$$

with $\widehat{dx_j} = dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_{n+1}$. The differential forms η_α satisfy

Proposition 2.5. The set of differential forms $\{\eta_\alpha\}_{\alpha \in I}$ restricted to L_b are a basis for the n-th de Rham comohology $H^n_{dR}(L_b)$ of L_b.

Proof. See [Mov20, Proposition 15.2, Theorem 10.1]. ■

In what follows we calculate the integral of the cycles and differential forms previously described. This is a reproduction of [Mov20, §15.2]. We will need the multi parameter version of the beta function.

$$\begin{aligned} B(a_1, a_2, \ldots, a_{n+1}) &:= \int_{\Delta^n} t_1^{a_1-1} \cdots t_{n+1}^{a_{n+1}-1} dt_1 \wedge \cdots \wedge dt_n \\ &= \frac{\Gamma(a_1) \cdots \Gamma(a_{n+1})}{\Gamma(a_1 + \cdots + a_{n+1})} \end{aligned} \tag{2.3}$$

$$Re(a_1), \ldots, Re(a_{n+1}) > 0,$$

where $\Gamma(t)$ is the Gamma function. The following proposition was first done by Deligne in [Del82] for the classical Fermat variety with $m_1 = \cdots = m_{n+1}$.

Proposition 2.6. Let $g = x_1^{m_1} + \cdots + x_{n+1}^{m_{n+1}}$, then

$$\int_{\Delta_{\alpha+a}} \eta_\beta = \Lambda B\left(\frac{\beta_1+1}{m_1}, \ldots, \frac{\beta_n+1}{m_n}, \frac{\beta_{n+1}+1}{m_{n+1}}\right),$$

where

$$\Lambda = (-1)^n \left(\prod_{j=1}^{n+1} \frac{1}{m_j}\right) \left(\prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta_j+1)(\alpha_j+a_j)}\right).$$

Therefore

$$\int_{\delta_\alpha} \eta_\beta = \frac{(-1)^n}{\prod_{j=1}^{n+1} m_j} \prod_{j=1}^{n+1} \left(\zeta_{m_j}^{(\alpha_j+1)(\beta_j+1)} - \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) B\left(\frac{\beta_1+1}{m_1}, \ldots, \frac{\beta_n+1}{m_n}, \frac{\beta_{n+1}+1}{m_{n+1}}\right).$$

Proof. First, note that $\eta_\beta = \frac{x^\beta dx}{dg}$. Therefore

$$\begin{aligned}\int_{\Delta_{\alpha+a}} \frac{x^\beta dx}{dg} =& \frac{(-1)^n}{m_{n+1}} \int_{\Delta_{\alpha+a}} x_1^{\beta_1} \cdots x_n^{\beta_n} x_{n+1}^{\beta_{n+1}-m_{n+1}+1} dx_1 \wedge \cdots \wedge dx_n \\ =& \frac{(-1)^n}{m_{n+1}} \prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta_j+1)(\alpha_j+a_j)} \times \\ & \int_{\Delta^n} t^{\beta_1/m_1} \cdots t^{\beta_n/m_n} (t^{1/m_{n+1}})^{\beta_{n+1}-m_{n+1}+1} d(t_1^{1/m_1}) \wedge \cdots \wedge d(t_n^{1/m_n}) \\ =& \lambda \int_{\Delta^n} t_1^{\frac{\beta_1+1}{m_1}-1} \cdots t_n^{\frac{\beta_n+1}{m_n}-1} t_{n+1}^{\frac{\beta_{n+1}+1}{m_{n+1}}-1} dt_1 \wedge \cdots \wedge dt_n \\ =& \lambda B\left(\frac{\beta_1+1}{m_1}, \cdots, \frac{\beta_n+1}{m_n}, \frac{\beta_{n+1}+1}{m_{n+1}}\right) \text{ By equation (2.3).}\end{aligned}$$

The above proves the first part. For the second it is sufficient to prove that

$$\sum_{a\in\{0,1\}^{n+1}} (-1)^{\sum_{i=1}^{n+1}(1-a_i)} \prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta_j+1)(\alpha_j+a_j)} = \prod_{j=1}^{n+1} \zeta_{m_j}^{(\beta_j+1)(\alpha_j+a_j)}.$$

This is obtained by a simple inductive argument. ∎

Remark 2.2. Via the biholomorphism ϕ_b the periods of L_b are given by

$$\int_{\delta_\alpha^b} \frac{x^\beta dx}{dg} = b^{\sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}-1} \int_{\delta_\alpha} \frac{x^\beta dx}{dg}. \tag{2.4}$$

With the notations in §2.1, the above is rewritten as

$$p(\{g=b\}, \beta, \delta_\alpha^b) = b^{\sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}-1} p(\{g=1\}, \beta, \delta_\alpha).$$

If $A_\beta = \frac{\gamma+1}{q}$ is easy to see that

$$\frac{p(\{g=-b\}, \beta, \delta_\alpha^{-b})}{p(\{z^q=b\}, \gamma, \delta_3)} = \frac{p(\{g=-1\}, \beta, \delta_\alpha^{-1})}{p(\{z^q=1\}, \gamma, \delta_3)}. \tag{2.5}$$

The above tells us that in the case of the Fermat variety this quotient is independent of $b \neq 0$.

CHAPTER 3

Hypergeometric periods

Until now we have seen how to calculate integrals of the residue of a form with pole of order one over a joint cycle. In this chapter we will give a tool to calculate the integrals of differential forms with pole of higher order. For this we will reduce the pole order of the differential form, which will be explained in §3.1. This is also know as Griffiths-Dwork method. We use another version of this method in the affine chart taken from [Mov20]. Section §3.2 is devoted to calculate periods using the theory exhibited so far. Last section provides a criterion for when a differential form in the affine part actually comes from a differential form in the compactification. This will be important to establish that certain hypergeometric periods are algebraic. The general theory of this chapter is developed for tame polynomials in [Mov20]. Here we will do more computations in our case $f(x, y) = g(x) + P(y)$ that are not included in [Mov20], where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$ and $P(y)$ is a polynomial with discriminant non zero.

3.1 Pole reduction

Throughout this chapter we use y and x_{n+1} interchangeably. Let $h(x) \in \mathbb{C}[x]$ be a polynomial in the variables $x = (x_1, \ldots, x_{n+1})$. We define the $\mathbb{C}$-module

$$V_h := \frac{\mathbb{C}[x]}{\langle \frac{\partial h}{\partial x_j} \rangle_{1 \leq j \leq n+1}},$$

called the Milnor module.

Definition 3.1. Let h be a tame polynomial. The multiplication by h induces a $\mathbb{C}$-linear map in V_h with associated matrix A. We define the **discriminant** of h to be $\Delta_h := det(A)$.

Let us consider $f = g + P$ with $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$ and $P = P(y)$ is a polynomial. Also consider the Milnor module

$$V_f := \frac{\mathbb{C}[x]}{\langle \frac{\partial f}{\partial x_j} \rangle_{j:\overline{1,n+1}}} = \frac{\mathbb{C}[x', y]}{\langle x_i^{m_i - 1}, dP \rangle_{i:\overline{1,n}}},$$

whose base is given by

$$\left\{x_1^{\beta_1}\dots x_n^{\beta_n}y^{\beta_{n+1}}|\ 0\le\beta_j\le m_j-2;1\le j\le n+1\right\}=\left\{x'^{\beta'}y^{\beta_{n+1}}|\ (\beta',\beta_{n+1})\in I\right\}.$$

Using this base we can represent the linear maps $V_f\xrightarrow{\cdot f}V_f$ and $V_P\xrightarrow{\cdot P}V_P$ by the matrices A and B. Note that

$$f\cdot x'^{\beta'}y^{\beta_{n+1}}=P\cdot x'^{\beta'}y^{\beta_{n+1}}\ \text{ in }\ V_f.$$

Thus the linear map induced by multiplication by f, with fixed β', in the submodule

$$\langle x'^{\beta'},x'^{\beta'}y,\dots,x'^{\beta'}y^{m-2}\rangle\subset V_f$$

has as matrix B. Therefore the linear map $V_f\xrightarrow{f}V_f$ has matrix

$$A=\begin{bmatrix}B&0&\cdots&0&0\\0&B&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\cdots&B&0\\0&0&\cdots&0&B\end{bmatrix}$$

with B repeated $N=\prod_{i=1}^{n}(m_i-1)$ times. We have

Proposition 3.1. $\Delta_f=\Delta_P^N$.

Proof. Note that $\Delta_P=det(B)$ and by the last expression of matrix A, the result is concluded. ■

From now on we will denote $\Delta:=\Delta_P$. In the case of one variable, the definition of discriminant given above coincides up to multiplication with a constant, with the definition of discriminant given in terms of the resultant. Thus, there are polynomials $Q_1(y)$, $Q_2(y)$ that

$$\Delta=Q_1\frac{\partial P}{\partial y}+PQ_2. \tag{3.1}$$

Example 3.1. Our main case of interest is when $P=y(1-y)(\lambda-y)$. In this case we have $\Delta=\lambda^2(1-\lambda)^2$. The polynomials Q_1, Q_2 satisfying equation (3.1) are given by

$$Q_1(y)=a_\lambda y^2+b_\lambda y+c_\lambda,\ \ Q_2(y)=-3a_\lambda y+e_\lambda,$$

with

$$a_\lambda=2(\lambda^2-\lambda+1),\ \ b_\lambda=-(2\lambda^3-\lambda^2-\lambda+2),$$
$$c_\lambda=\lambda(1-\lambda)^2,\ \ e_\lambda=4\lambda^3-3\lambda^2-3\lambda+4.$$

The following description of the differential form Δdx will help us to reduce the pole order of a differential form with a pole along $\{f=0\}$.

Proposition 3.2. There is a n-form ξ

$$\Delta dx = df \wedge \xi + fQ_2 dx, \tag{3.2}$$

with $Q_2(y)$ as above.

Proof. First remember that $x = (x', y)$ and $dx = dx_1 \wedge \cdots \wedge dx_n \wedge dy$. Consider

$$\xi = Q_1(y)dx' - Q_2(y)\eta' \wedge dy,$$

where $\eta' := \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{m_i} \widehat{dx'_i}$, with $\widehat{dx'_i} := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$. ■

From equality (3.2), it follows that there are n-forms ξ_β such that

$$\Delta \omega_\beta = df \wedge \xi_\beta + fQ_2\omega_\beta,$$

namely $\xi_\beta = x^\beta \xi$. Thus

$$\Delta \frac{\omega_\beta}{f^j} = \frac{df \wedge \xi_\beta + fQ_2\omega_\beta}{f^j} = \frac{1}{j-1}\left(\frac{d\xi_\beta}{f^{j-1}} - d\left(\frac{\xi_\beta}{f^{j-1}}\right)\right) + \frac{Q_2\omega_\beta}{f^{j-1}}.$$

Using that $d(x'^{\beta'}\eta') = A_{\beta'} x'^{\beta'} dx'$, we have

$$\begin{aligned} d\xi_\beta &= x'^{\beta'}\left(\beta_{n+1} y^{\beta_{n+1}-1} Q_1(y) + y^{\beta_{n+1}} \frac{\partial Q_1(y)}{\partial y}\right) dx' \wedge dy - A_{\beta'} Q_2(y)\omega_\beta \\ &= \beta_{n+1} Q_1 \omega_{\beta-(0',1)} + (Q'_1 - A_{\beta'} Q_2)\omega_\beta, \end{aligned}$$

where $\beta = (\beta', \beta_{n+1})$, $(0', 1) \in I$ and $A_{\beta'} := \sum_{i=1}^{n} \frac{\beta_i + 1}{m_i}$. Therefore in $H_{dR}^{n+1}(\mathbb{C}^{n+1} \backslash U)$ we obtain the following formula that allows us to reduce the pole order:

$$\begin{aligned} \left[\frac{\omega_\beta}{f^j}\right] &= \frac{1}{\Delta}\left[\frac{1}{j-1}\left(\frac{d\xi_\beta}{f^{j-1}}\right) + \frac{Q_2\omega_\beta}{f^{j-1}}\right] \\ &= \frac{1}{\Delta}\left[\frac{\beta_{n+1}Q_1\omega_{\beta-(0',1)} + (Q'_1 - A_{\beta'}Q_2)\omega_\beta}{(j-1)f^{j-1}} + \frac{Q_2\omega_\beta}{f^{j-1}}\right] \\ &= \frac{1}{\Delta}\left[\frac{\beta_{n+1}Q_1}{j-1}\frac{\omega_{\beta-(0',1)}}{f^{j-1}} + \left(\left(1 - \frac{A_{\beta'}}{j-1}\right)Q_2 + \frac{Q'_1}{j-1}\right)\frac{\omega_\beta}{f^{j-1}}\right] \end{aligned} \tag{3.3}$$

Remark 3.1. If $\beta = (\beta', 0)$ then

$$\left[\frac{\omega_\beta}{f^j}\right] = \frac{1}{\Delta^{j-1}(j-1)!}\prod_{k=1}^{j-1}\left[\left((k - A_{\beta'})Q_2 + Q'_1\right)\right]\frac{\omega_\beta}{f}$$

Proposition 3.3. Consider

$$C_k := \frac{1}{\Delta}\left(\frac{Q'_1(y)}{k} + \left(1 - \frac{A_{\beta'}}{k}\right)Q_2(y)\right),$$

$$D_k^j := \frac{C_1 C_2 \dots C_{k-1} C_{k+1} \dots C_j}{k}.$$

Then for $j \geq 3$ and $\beta_{n+1} = 0, 1, 2$, we have

$$\begin{aligned}\left[\frac{\omega_\beta}{f^j}\right] =& \frac{\beta_{n+1}(\beta_{n+1}-1)Q_1^2}{\Delta^2}\left(\frac{1}{j-1}\sum_{k=1}^{j-2} D_k^{j-2} + \sum_{l=2}^{j-2}\frac{1}{j-l}\prod_{k=j-l+1}^{j-1} C_k \sum_{k=1}^{j-l-1} D_k^{j-l-1}\right)\frac{\omega_{(\beta',\beta_{n+1}-2)}}{f} \\ &+ \frac{\beta_{n+1}Q_1}{\Delta}\left(\sum_{k=1}^{j-1} D_k^{j-1}\right)\frac{\omega_{(\beta',\beta_{n+1}-1)}}{f} + \prod_{k=1}^{j-1} C_k \frac{\omega_\beta}{f}.\end{aligned}$$

Proof. The above remark is just the case when $\beta_{n+1} = 0$. Let us look when $\beta_{n+1} = 1$ and the other case is analogous. By pole order reduction and induction in the order of the pole, we have

$$\begin{aligned}\left[\frac{\omega_\beta}{f^j}\right] &= \frac{Q_1}{\Delta(j-1)}\frac{\omega_{(\beta',0)}}{f^{j-1}} + C_{j-1}\frac{\omega_\beta}{f^{j-1}} \\ &= \frac{Q_1}{\Delta(j-1)}\prod_{k=1}^{j-2} C_k \frac{\omega_{(\beta',0)}}{f} + C_{j-1}\left(\frac{Q_1}{\Delta}\left(\sum_{k=1}^{j-2} D_k^{j-2}\right)\frac{\omega_{(\beta',0)}}{f} + \prod_{k=1}^{j-2} C_k\frac{\omega_\beta}{f}\right) \\ &= \frac{Q_1}{\Delta}\left(\sum_{k=1}^{j-1} D_k^{j-1}\right)\frac{\omega_{(\beta',0)}}{f} + \prod_{k=1}^{j-1} C_k \frac{\omega_\beta}{f}\end{aligned}$$

■

3.2 Calculating periods

Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of f. Let the affine part $U := \{f = 0\} \subset \mathbb{C}^{n+1}$ where $f = g(x) + P(y)$, $g(x) = x_1^{m_1} + x_2^{m_2} + \dots + x_n^{m_n}$, $m_j \geq 2$ and $P(y) : \mathbb{C} \to \mathbb{C}$ is a polynomial of fixed degree m and with 1 as a regular value.

Let $\alpha \in J := \prod_{j=1}^n I_{m_j}$, $\delta_\alpha^{-b} \in H_n(\{g(x) = -b\}, \mathbb{Z})$ defined as in §2.3. Let $\delta_1 \in H_0(\{P(y) = b\}, \mathbb{Z})$ be a vanishing cycle and t_s, $s \in [0, 1]$ be a path in the $\mathbb{C}$-plane which connects a critical value of P to 0 (the unique critical value of g). We assume that δ_1 vanishes along t^{-1} and δ_α^{-b} vanishes along t and $b \in t$ regular value of $P(y)$.

$$\begin{aligned}
\int_{\delta_1 *_t \delta_\alpha^{-b}} res\left(\frac{\omega_\beta}{f}\right) &= \int_{\delta_1 *_t \delta_\alpha^{-b}} \frac{\omega_\beta}{df} \\
&= \int_{\delta_1 *_t \delta_\alpha^{-b}} \frac{(y^{\beta_{n+1}}dy) \wedge (x'^{\beta'}dx')}{d(P+g)} \\
&= \frac{p(\{g=-b\}, \beta', \delta_\alpha^{-b})}{p(\{z^q=b\}, \gamma, \delta_3)} \int_{\delta_1 *_t \delta_3} \frac{y^{\beta_{n+1}} z^\gamma dy \wedge dz}{d(P - z^q)} \quad \text{By Proposition 2.2.} \\
&= \frac{p(\{g=-b\}, \beta', \delta_\alpha^{-b})}{q \cdot p(\{z^q=b\}, \gamma, \delta_3)} \int_{\delta_1 *_t \delta_3} y^{\beta_{n+1}} z^{\gamma-q+1} dy \\
&= \frac{p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} \int_C y^{\beta_{n+1}} P(y)^{\frac{\gamma-q+1}{q}} dy. \text{ By (2.5).}
\end{aligned}$$

Where $\delta_3 = b^{1/q}[\zeta_q - 1] \in H_0(\{z^q = b\}, \mathbb{Z})$, with q and γ given by the equality $A_{\beta'} = \frac{\gamma+1}{q}$ and $C : [0,1] \to \mathbb{C}$ is the path induced by the projection of the joint cycle $\delta_1 *_t \delta_3$ in the y-coordinate.

Remark 3.2. If there is another path $\hat{t}_s$ that connects the same critical value of $P(y)$ with 0 such that $1 \in \hat{t}_s$ and when deforming it to path t_s does not cross other critical values, we have

$$\int_{\delta_1 *_t \delta_\alpha^{-b}} res\left(\frac{\omega_\beta}{f}\right) = \int_{\delta_1 *_{\hat{t}} \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f}\right),$$

and the same is true of higher-order pole forms. That is why in what follows we will always put in the integration domain the cycle $\delta_1 *_t \delta_\alpha^{-1}$.

Consider $P(y) = y(y-1)(y-\lambda)$ with $\lambda \in \mathbb{R}$ and $\lambda > 1$. Let c_1 be the critical value obtained by the evaluation in $P(y)$ of the critical point between roots 0 and 1, and $t : [0,1] \to \mathbb{C}$; $t(s) = c_1(1-s)$ with $s \in [0,1]$. Let $\delta_{0t} := [y_{1t} - y_{2t}] \in H_0(\{P(y) = t\}, \mathbb{Z})$ be vanishing cycle along t_s^{-1}, which satisfies that $\delta_{0t(1)} = [1] - [0]$. Let $\delta_{3t} := t^{1/q}[\zeta_q - 1] \in H_0(\{z^q = t\}, \mathbb{Z})$ be vanishing cycle along t_s. In this context we have

$$\begin{aligned}
\int_{\delta_0 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f}\right) &= \frac{p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} \int_{\delta_0 *_t \delta_3} y^{\beta_{n+1}} z^{\gamma-q+1} dy \\
&= \frac{p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} \int_C y^{\beta_{n+1}} P(y)^{\frac{\gamma-q+1}{q}} dy \\
&= \frac{p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} \int_0^1 s^{\beta_{n+1}} [s(s-1)(s-\lambda)]^{\frac{\gamma-q+1}{q}} ds \\
&= \frac{\lambda^{\frac{\gamma-q+1}{q}} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} \int_0^1 s^{\beta_{n+1}} [s(1-s)(1-\frac{s}{\lambda})]^{\frac{\gamma-q+1}{q}} ds \\
&= \frac{\lambda^{\frac{\gamma-q+1}{q}} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q=1\}, \gamma, \delta_3)} B\left(\frac{\gamma+1}{q} + \beta_{n+1}, \frac{\gamma+1}{q}\right) \cdot \\
&\quad F\left(\frac{\gamma+1}{q} + \beta_{n+1}, -\frac{\gamma-q+1}{q}, \frac{2(\gamma+1)}{q} + \beta_{n+1}; \frac{1}{\lambda}\right)
\end{aligned}$$

where equation (A.2) was used in the last equality.

Now if c_2 is the other critical value of $P(y)$. Let $\delta_{1t} := [y_{1t} - y_{2t}] \in H_0(\{P(y) = t\}, \mathbb{Z})$ be vanishing cycle along t_s^{-1} with $t : [0,1] \to \mathbb{C}$; $t(s) = c_2(1-s)$; $s \in [0,1]$ satisfying that $\delta_{1t(1)} = [\lambda] - [1]$ and again let $\delta_{3t} := t^{1/q}[\zeta_q - 1] \in H_0(\{z^q = t\}, \mathbb{Z})$ be vanishing cycle along t_s. We have

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f}\right) &= \frac{p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q = 1\}, \gamma, \delta_3)} \int_C y^{\beta_{n+1}} P(y)^{\frac{\gamma - q + 1}{q}} dy \\ &= \frac{(\lambda - 1) p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q = 1\}, \gamma, \delta_3)} \cdot \\ &\quad \int_0^1 (\lambda s + 1 - s)^{\beta_{n+1} + \frac{\gamma - q + 1}{q}} [(\lambda s - s)(\lambda s + 1 - s - \lambda)]^{\frac{\gamma - q + 1}{q}} ds \\ &= \frac{(-1)^{\frac{\gamma - q + 1}{q}} (\lambda - 1)^{2\frac{\gamma - q + 1}{q} + 1} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q = 1\}, \gamma, \delta_3)} \cdot \\ &\quad \int_0^1 [1 - (1-\lambda)s]^{\beta_{n+1} + \frac{\gamma - q + 1}{q}} [s(1-s)]^{\frac{\gamma - q + 1}{q}} ds \\ &= \frac{(-1)^{\frac{\gamma - q + 1}{q}} (\lambda - 1)^{2\frac{\gamma - q + 1}{q} + 1} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{q \cdot p(\{z^q = 1\}, \gamma, \delta_3)} B\left(\frac{\gamma + 1}{q}, \frac{\gamma + 1}{q}\right) \cdot \\ &\quad F\left(\frac{\gamma + 1}{q}, -\frac{\gamma - q + 1}{q} - \beta_{n+1}, \frac{2(\gamma + 1)}{q}; 1 - \lambda\right).\end{aligned}$$

Remark 3.3. The explicit calculations previously made are still valid if λ is in a complex neighborhood of $Re(\lambda) > 1$.

Remember that $p(\{z^q = 1\}, \gamma, \delta_3) = \frac{\zeta_q^{\gamma+1} - 1}{q}$. The previous calculations are recorded in the following proposition.

Proposition 3.4. Let $f = g(x) + P(y)$ be a polynomial with $g(x) = x_1^{m_1} + x_2^{m_2} + \cdots + x_n^{m_n}$, $m_j \geq 2$ and $P(y) = y(1-y)(\lambda - y)$, $\lambda > 1$. Consider the cycles $\delta_0 := [1] - [0], \delta_1 := [\lambda] - [1] \in H_0(\{P(y) = 0\}, \mathbb{Z})$ and t_s the straight line connecting one of the critical values of $P(y)$ with 0. If $A_{\beta'} \notin \mathbb{N}$, we have

$$\begin{aligned}\int_{\delta_0 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f}\right) &= \frac{\lambda^{A_{\beta'} - 1} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{\zeta_q^{\gamma+1} - 1} B\left(A_{\beta'} + \beta_{n+1}, A_{\beta'}\right) \cdot \\ &\quad F\left(A_{\beta'} + \beta_{n+1}, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1}; \frac{1}{\lambda}\right).\end{aligned} \tag{3.4}$$

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f}\right) &= \frac{(-1)^{A_{\beta'}} (1-\lambda)^{2A_{\beta'} - 1} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{\zeta_q^{\gamma+1} - 1} B\left(A_{\beta'}, A_{\beta'}\right) \cdot \\ &\quad F\left(A_{\beta'}, 1 - A_{\beta'} - \beta_{n+1}, 2A_{\beta'}; 1 - \lambda\right),\end{aligned} \tag{3.5}$$

where

$$p(\{g=-1\},\beta',\delta_\alpha^{-1}) = \frac{(-1)^{n+\sum_{j=1}^n \frac{\beta_j+1}{m_j}}}{\prod_{j=1}^n m_j}\prod_{j=1}^n \left(\zeta_{m_j}^{(\alpha_j+1)(\beta_j+1)} - \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) B\left(\frac{\beta_1+1}{m_1},\dots,\frac{\beta_n+1}{m_n}\right).$$

Now we want to calculate

$$\int_{\delta_0 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right), \quad \int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right).$$

For this, we use pole order reduction (equality (3.3)), and repeating the previous calculation we have

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right) &= \frac{1}{\Delta}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left[\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)\frac{\omega_\beta}{f} + \beta_{n+1}Q_1(y)\frac{\omega_{\beta-(0,1)}}{f}\right]\\ &= \frac{1}{\Delta}\int_{\delta_1 *_t \delta_\alpha^{-1}} \left[\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)\frac{\omega_\beta}{df} + \beta_{n+1}Q_1(y)\frac{\omega_{\beta-(0,1)}}{df}\right]\\ &= \frac{p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta p(\{z^q=1\},\gamma,\delta_3)}\left[\int_{\delta_1 *_t \delta_3} \frac{\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)y^{\beta_{n+1}}z^\gamma dy\wedge dz}{d(P-z^q)}\right.\\ &\left. + \beta_{n+1}\int_{\delta_1 *_t \delta_3}\frac{Q_1(y)y^{\beta_{n+1}-1}z^\gamma dy\wedge dz}{d(P-z^q)}\right] \quad \text{By Proposition 2.2.}\end{aligned}$$

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right) &= \frac{p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta p(\{z^q=1\},\gamma,\delta_3)}\cdot\\ &\int_{\delta_1 *_t \delta_3} \frac{\Big(\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)y+\beta_{n+1}Q_1(y)\Big)y^{\beta_{n+1}-1}z^\gamma dy\wedge dz}{d(P-z^q)}\\ =&\frac{p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta q\cdot p(\{z^q=1\},\gamma,\delta_3)}\cdot\\ &\int_{\delta_1 *_t \delta_3}\Big(\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)y+\beta_{n+1}Q_1(y)\Big)y^{\beta_{n+1}-1}z^{\gamma-q+1}dy\\ =&\frac{p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta q\cdot p(\{z^q=1\},\gamma,\delta_3)}\cdot\\ &\int_C\Big(\big((1-A_{\beta'})Q_2(y)+Q_1'(y)\big)y+\beta_{n+1}Q_1(y)\Big)y^{\beta_{n+1}-1}P(y)^{\frac{\gamma-q+1}{q}}dy,\end{aligned} \tag{3.6}$$

where C is the path induced by the projection of the joint cycle $\delta_1 *_t \delta_3$ in the y-coordinate. In the above, we could have used Proposition 2.3 instead of Proposition 2.2. Let us go back to our case of interest when $P=y(1-y)(\lambda-y)$. To continue our computations we need to know Δ, Q_1, Q_2 that satisfy equation (3.1). From Example 3.1 let us remember

$$\Delta=\lambda^2(1-\lambda)^2,\ \ Q_1(y)=a_\lambda y^2+b_\lambda y+c_\lambda,\ \ Q_2(y)=-3a_\lambda y+e_\lambda,$$

with

$$a_\lambda = 2(\lambda^2 - \lambda + 1), \quad b_\lambda = -(2\lambda^3 - \lambda^2 - \lambda + 2),$$

$$c_\lambda = \lambda(1-\lambda)^2, \quad e_\lambda = 4\lambda^3 - 3\lambda^2 - 3\lambda + 4.$$

Thus using equation (3.6) and calculating we get

Proposition 3.5. In the same context of Proposition 3.4, we have

$$\begin{aligned}\int_{\delta_0 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right) =& \frac{\lambda^{A_{\beta'}-3} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{(1-\lambda)^2(\zeta_q^{\gamma+1}-1)} B\left(A_{\beta'} + \beta_{n+1} - 1, A_{\beta'}\right) \times \\ & \left[\frac{(A_{\beta'} + \beta_{n+1} - 1)_2}{(2A_{\beta'} + \beta_{n+1} - 1)_2} F\left(A_{\beta'} + \beta_{n+1} + 1, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1} + 1; \frac{1}{\lambda}\right) \times \right. \\ & \left(3A_{\beta'} + \beta_{n+1} - 1\right) a_\lambda + \\ & \frac{A_{\beta'} + \beta_{n+1} - 1}{2A_{\beta'} + \beta_{n+1} - 1} F\left(A_{\beta'} + \beta_{n+1}, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1}; \frac{1}{\lambda}\right) \times \\ & \quad \left((1 - A_{\beta'})e_\lambda + (1 + \beta_{n+1})b_\lambda\right) \\ & \left. + F\left(A_{\beta'} + \beta_{n+1} - 1, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1} - 1; \frac{1}{\lambda}\right) \beta_{n+1} c_\lambda \right].\end{aligned}$$

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right) =& \frac{(-1)^{A_{\beta'}-1}(\lambda-1)^{2A_{\beta'}-3} p(\{g=-1\}, \beta', \delta_\alpha^{-1})}{\lambda^2(\zeta_q^{\gamma+1}-1)} B(A_{\beta'}, A_{\beta'}) \times \\ & \left[F\left(A_{\beta'}, -(A_{\beta'} + \beta_{n+1}), 2A_{\beta'}; 1 - \lambda\right) \left(3A_{\beta'} + \beta_{n+1} - 1\right) a_\lambda + \right. \\ & F\left(A_{\beta'}, -(A_{\beta'} + \beta_{n+1} - 1), 2A_{\beta'}; 1 - \lambda\right) \left((1 - A_{\beta'})e_\lambda + (1 + \beta_{n+1})b_\lambda\right) \\ & \left. + F\left(A_{\beta'}, -(A_{\beta'} + \beta_{n+1} - 2), 2A_{\beta'}; 1 - \lambda\right) \beta_{n+1} c_\lambda \right],\end{aligned}$$

with

$$p(\{g=-1\}, \beta', \delta_\alpha^{-1}) = \frac{(-1)^{n+A_{\beta'}}}{\prod_{j=1}^n m_j} \prod_{j=1}^n \left(\zeta_{m_j}^{(\alpha_j+1)(\beta_j+1)} - \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) B\left(\frac{\beta_1+1}{m_1}, \ldots, \frac{\beta_n+1}{m_n}\right).$$

To illustrate a little more, let us see the result of the integral on a differential form with a pole of order 3.

Proposition 3.6. In the same context of Proposition 3.4, we have

$$\int_{\delta_0 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^3}\right) = \frac{\lambda^{A_{\beta'}-1} p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta^2 q \cdot p(\{z^q=1\},\gamma,\delta_3)} \times$$
$$\Bigg[D_1 B(A_{\beta'}+\beta_{n+1}+2, A_{\beta'}) F\left(A_{\beta'}+\beta_{n+1}+2, 1-A_{\beta'}, 2A_{\beta'}+\beta_{n+1}+2; \frac{1}{\lambda}\right) +$$
$$D_2 B(A_{\beta'}+\beta_{n+1}+1, A_{\beta'}) F\left(A_{\beta'}+\beta_{n+1}+1, 1-A_{\beta'}, 2A_{\beta'}+\beta_{n+1}+1; \frac{1}{\lambda}\right) +$$
$$D_3 B(A_{\beta'}+\beta_{n+1}, A_{\beta'}) F\left(A_{\beta'}+\beta_{n+1}, 1-A_{\beta'}, 2A_{\beta'}+\beta_{n+1}; \frac{1}{\lambda}\right) +$$
$$D_4 B(A_{\beta'}+\beta_{n+1}-1, A_{\beta'}) F\left(A_{\beta'}+\beta_{n+1}-1, 1-A_{\beta'}, 2A_{\beta'}+\beta_{n+1}-1; \frac{1}{\lambda}\right)\Bigg], \tag{3.7}$$

.

$$\begin{aligned}\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^3}\right) = \frac{(-1)^{A_{\beta'}-1}(\lambda-1)^{2A_{\beta'}-1} p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta^2 q \cdot p(\{z^q=1\},\gamma,\delta_3)} B(A_{\beta'},A_{\beta'}) \times \\ \big[D_1 F\left(A_{\beta'}, -(A_{\beta'}+\beta_{n+1}+1), 2A_{\beta'}; 1-\lambda\right) + \\ D_2 F\left(A_{\beta'}, -(A_{\beta'}+\beta_{n+1}), 2A_{\beta'}; 1-\lambda\right) + \\ D_3 F\left(A_{\beta'}, -(A_{\beta'}+\beta_{n+1}-1), 2A_{\beta'}; 1-\lambda\right) + \\ D_4 F\left(A_{\beta'}, -(A_{\beta'}+\beta_{n+1}-2), 2A_{\beta'}; 1-\lambda\right)\big],\end{aligned} \tag{3.8}$$

where

$$p(\{g=-1\},\beta',\delta_\alpha^{-1}) = \frac{(-1)^{n+\sum_{j=1}^n \frac{\beta_j+1}{m_j}}}{\prod_{j=1}^n m_j} \prod_{j=1}^n \left(\zeta_{m_j}^{(\alpha_j+1)(\beta_j+1)} - \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) B\left(\frac{\beta_1+1}{m_1},\ldots,\frac{\beta_n+1}{m_n}\right),$$

$$D_1 = \left(9A_{\beta'}^2 + 3A_{\beta'} - 2 - \beta_{n+1}(6A_{\beta'}-13)\right) a_\lambda,$$
$$D_2 = \left(-6A_{\beta'}^2 - 6A_{\beta'}b_\lambda + 22A_{\beta'}e_\lambda + 13b_\lambda - 18e_\lambda - \beta_{n+1}(6A_{\beta'}b_\lambda - 2A_{\beta'}e_\lambda - 15b_\lambda + 3e_\lambda)\right) a_\lambda,$$
$$\begin{aligned} D_3 = {} & \left(A_{\beta'}^2 e_\lambda + 2A_{\beta'}b_\lambda - 3A_{\beta'}e_\lambda - 3b_\lambda + 2e_\lambda\right) e_\lambda + b_\lambda \\ & - \beta_{n+1}\left(6A_{\beta'}a_\lambda c_\lambda + 13a_\lambda c_\lambda - 2A_{\beta'}b_\lambda e_\lambda - 2b_\lambda^2 + 3b_\lambda e_\lambda\right),\end{aligned}$$
$$D_4 = -\beta_{n+1}\left(3e_\lambda - 2A_{\beta'}e_\lambda - 2b_\lambda\right) c_\lambda.$$

Remark 3.4. Any such integral is expressed

$$\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^{j+1}}\right) = \frac{p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\Delta^j q \cdot p(\{z^q=1\},\gamma,\delta_3)} \int_C R(y,\beta) y^{\beta_{n+1}-j} P(y)^{\frac{\gamma-q+1}{q}} dy, \tag{3.9}$$

with $R(y,\beta)$ is a polynomial obtained by pole order reduction.

3.3 Pole increment

We start this section by giving a criterion for when a differential form in the affine part actually comes from a differential form in the compactification. This criterion can be found in [Mov20, Proposition 11.4] or [Ste77, Lemma 2].

Proposition 3.7. If $A_\beta = k \in \mathbb{N}$, the meromorphic form $\frac{\omega_\beta}{f^k}$ has pole of order one at infinity. If $A_\beta < k \in \mathbb{N}$, the meromorphic form $\frac{\omega_\beta}{f^k}$ has no pole at infinity.

Proof. First, note that $d\left(\frac{x_j}{x_0^{v_j}}\right) = x_0^{-v_j}dx_j - v_j x_j x_0^{-v_j-1}dx_0$. Thus

$$\begin{aligned}\frac{\omega_\beta}{f^k} &= \frac{\left(\frac{x_1}{x_0^{v_1}}\right)^{\beta_1}\cdots\left(\frac{x_{n+1}}{x_0^{v_{n+1}}}\right)^{\beta_{n+1}} d\left(\frac{x_1}{x_0^{v_1}}\right)\wedge\cdots\wedge d\left(\frac{x_{n+1}}{x_0^{v_{n+1}}}\right)}{f\left(\frac{x_1}{x_0^{v_1}},\ldots,\frac{x_{n+1}}{x_0^{v_{n+1}}}\right)^k}\\ &= \frac{Nx^\beta\eta_{(1,v)}}{x_0^{1-kN+\sum_{j=1}^{n+1}(\beta_j+1)v_j}\left(x_0\widehat{f}+f_N(x_1,\ldots,x_{n+1})\right)^k}\\ &= \frac{Nx^\beta\eta_{(1,v)}}{x_0^{1+(A_\beta-k)N}\left(x_0\widehat{f}+f_N(x_1,\ldots,x_{n+1})\right)^k},\end{aligned}$$

with $\widehat{f} = x_0^{N-1}f_0+x_0^{N-2}f_1+\cdots+x_0f_{N-2}+f_{N-1}$, where $f=\sum_{i=0}^N f_i$ with f_i a v-weighted homogeneous polynomial of degree i, $f_N \neq 0$ and

$$\eta_{(1,v)} = \sum_{j=0}^{n+1}(-1)^j\frac{v_j}{N}x_j\widehat{dx_j}.$$

If $A_\beta = k \in \mathbb{N}$ follows from the previous equation that the meromorphic form $\frac{\omega_\beta}{f^k}$ has pole of order one at infinity, likewise if $A_\beta < k \in \mathbb{N}$, $\frac{\omega_\beta}{f^k}$ has not pole at infinity. ■

How to know if the meromorphic form $\frac{\omega_\beta}{f^k}$, $A_\beta > k$ comes from a form in the compactification? For this, we increment the pole order and apply the previous proposition. To increment the pole order we reproduce [Mov20, Proposition 11.3]. Consider $\eta := \sum_{i=1}^{n+1}(-1)^{i-1}\frac{x_i}{m_i}\widehat{dx_i}$ and $\eta_\beta = x^\beta\eta$. Observe that $d\eta_\beta = A_\beta\omega_\beta$, thus

$$\begin{aligned}\frac{\omega_\beta}{f^k} &= \frac{d\eta_\beta}{A_\beta f^k}\\ &= \frac{1}{A_\beta}\left(\frac{kdf\wedge\eta_\beta}{f^{k+1}}+d\left(\frac{\eta_\beta}{f^k}\right)\right).\end{aligned}$$

Therefore in $H_{dR}^{n+1}(\mathbb{C}^{n+1}\backslash U)$ we have

$$\left[\frac{\omega_\beta}{f^k}\right] = \frac{k}{A_\beta}\left[\frac{df\wedge\eta_\beta}{f^{k+1}}\right]$$
$$= \frac{k}{A_\beta}\left[\frac{f\omega_\beta + (h-f)\omega_\beta + d(f-h)\wedge\eta_\beta}{f^{k+1}}\right]$$

and thus

$$\left[\frac{\omega_\beta}{f^k}\right] = \frac{A_\beta}{A_\beta - k}\left[\frac{(h-f)\omega_\beta + d(f-h)\wedge\eta_\beta}{f^{k+1}}\right], \tag{3.10}$$

where h is the last weighted homogeneous piece of f and satisfies $dh\wedge\eta_\beta = h\omega_\beta$. In our case $h = g(x) + y^{deg(P(y))} = x_1^{m_1} + \cdots + x_n^{m_n} + y^{deg(P(y))}$. If $P(y) = \sum_{j=0}^{m_{n+1}} c_j y^j$ with $c_{m_{n+1}} \neq 0$ we have

$$\left[\frac{\omega_\beta}{f^k}\right] = \frac{A_\beta}{m_{n+1}(A_\beta - k)}\left[\frac{\sum_{j=0}^{m_{n+1}-1}(j - m_{n+1})c_j\omega_{\beta+(0,j)}}{f^{k+1}}\right].$$

Therefore, using the process of pole order increment, the meromorphic form $\frac{\omega_\beta}{f^k}$ with $A_\beta > k$ can be written as a finite sum

$$\left[\frac{\omega_\beta}{f^k}\right] = \sum C_j\left[\frac{\omega_{\beta^{l_j}}}{f^j}\right] \tag{3.11}$$

with $A_{\beta^{l_j}} \leq j$, $k < j$, $\beta^{l_j} = (\beta_1^{l_j}, \ldots, \beta_{n+1}^{l_j})$ and $C_j \in \mathbb{C}$. Note that even when $A_{\beta^{l_j}} < j$ we can increment the pole order. We will stop the process of pole order increment the first time $A_{\beta^{l_j}} \leq j$ is satisfied.

Remark 3.5. For the polynomial $P(y) = y(y-1)(y-\lambda)$ the pole order increment looks like

$$\left[\frac{\omega_\beta}{f^k}\right] = \frac{A_\beta}{3(A_\beta - k)}\left[\frac{(1+\lambda)\omega_{\beta+(0',2)} - 2\lambda\omega_{\beta+(0',1)}}{f^{k+1}}\right].$$

Therefore, we can write the meromorphic form $\frac{\omega_\beta}{f^k}$ with $A_\beta > k$ as a finite sum

$$\left[\frac{\omega_\beta}{f^k}\right] = \sum C_j\left[\frac{\omega_{\beta+(0',k_j)}}{f^j}\right] \tag{3.12}$$

with $k < j$ and it is the first time that $A_{\beta+(0',k_j)} \leq j$. This means that $A_{\beta+(0',k_j)} \leq j$ and one step before reaching (3.12), the differential form $\frac{\omega_{\beta+(0',k_j-1)}}{f^{j-1}}$ appears with $j-1 < A_{\beta+(0',k_j-1)}$ or the differential form $\frac{\omega_{\beta+(0',k_j-2)}}{f^{j-1}}$ appears with $j-1 < A_{\beta+(0',k_j-2)}$.

Definition 3.2. A meromorphic form $\frac{\omega_\beta}{f^k}$ is called **good form** if $A_\beta < k$ or if $k < A_\beta \notin \mathbb{N}$ and the meromorphic form written as in equation (3.11), satisfies $A_{\beta^{l_j}} < j$.

Observe that a good form has no residue at infinity and hence it gives an element in $H^n_{dR}(X)$.

Example 3.2. Consider $f = g + P$ with $g(x) = x_1^2 + x_2^9$ and $P(y) = y^3 + ay^2 + by + c$. The meromorphic forms $\frac{\omega_\beta}{f}$ with $\beta = (0, \beta_2, \beta_3)$, $\beta_2 = 2, 5$ and $\beta_3 = 0, 1$ are good forms. Let us see it for $\beta_2 = 2$, we can write

$$\left[\frac{\omega_\beta}{f}\right] = \frac{A_\beta}{3(A_\beta - 1)}\left[\frac{-a\omega_{\beta+(0',2)} - 2b\omega_{\beta+(0',1)} - 3c\omega_\beta}{f^2}\right],$$

where $0' = (0, 0)$. If $\beta_3 = 0$ then $1 < A_\beta, A_{\beta+(0',1)}, A_{\beta+(0',2)} < 2$ and the above equation corresponds to equation (3.11). If $\beta_3 = 1$, then $1 < A_\beta, A_{\beta+(0',1)} < 2$ and $A_{\beta+(0',2)} > 2$. Therefore we apply again the pole order increment to the meromorphic form $\frac{\omega_{\beta+(0',2)}}{f^2}$ and we obtain

$$\begin{aligned}\left[\frac{\omega_\beta}{f}\right] =& \frac{-aA_\beta(A_\beta + 2/3)}{9(A_\beta - 1)(A_\beta - 4/3))}\left[\frac{-a\omega_{\beta+(0',4)} - 2b\omega_{\beta+(0',3)} - 3c\omega_{\beta+(0',2)}}{f^3}\right] + \\ & \frac{A_\beta}{3(A_\beta - 1)}\left[\frac{-2\omega_{\beta+(0',1)} - 3c\omega_\beta}{f^2}\right].\end{aligned}$$

with $2 < A_{\beta+(0',j)} < 3$, $j = 2, 3, 4$.

CHAPTER 4

Families of varieties and Hodge cycles

This chapter is about generic Hodge cycles and some applications. In §4.1 we will introduce this concept that capture those Hodge cycles that remain Hodge when doing monodromy along any path in a family of varieties. We will consider a subspace of the generic Hodge cycles that allows an elementary arithmetic definition, these are namely strong generic Hodge cycles. In §4.2 we will give an upper bound of the dimension of the strong generic Hodge cycles in some cases. In §4.3 we will briefly describe the Hodge numbers and we will give some formulas for them. In §4.4 we will find algebraic expressions involving hypergeometric functions using what has been developed so far. Finally, in §4.5 we will computationally verify the algebraicity of the expressions found in §4.4.

4.1 Generic and strong generic Hodge cycles

Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$, $P(y) : \mathbb{C} \to \mathbb{C}$ is a polynomial of degree $m = m_{n+1}$. Let

$$T := \left\{ t = (t_0, \ldots, t_m) \in \mathbb{C}^{m+1} \middle| t_m = 1, \ \ \Delta(P_t) \neq 0 \text{ where } P_t := \sum_{i=0}^{m} t_i y^i \right\}$$

be the space of polynomials of degree m with nonzero discriminant, and let

$$\mathcal{U} := \{(x, y, t) \in \mathbb{C}^n \times \mathbb{C} \times T |\ f_t(x, y) := g(x) + P_t(y) = 0\}$$

be the family of affine varieties parameterized by T. Thus the projection $\pi : \mathcal{U} \longrightarrow T$ is a locally trivial C^∞ fibration (see [Mov20, §7.4] and the references therein). We denote by $U_t := \pi^{-1}(t) = \{f_t = 0\} \subset \mathbb{C}^{n+1}$ and by X_t a desingularization of $D_t := \{F_t = 0\} \subset \mathbb{P}^{(1,v)}$ where F_t is the quasi-homogenization of f_t.

Definition 4.1. Fix $t_0 \in T$, we say that $\delta_{t_0} \in H_n(U_{t_0}, \mathbb{Q})$ is a **generic Hodge cycle** if δ_t is a Hodge cycle of X_t, this means, $\delta_t \in \mathrm{Hod}_n(X_t, \mathbb{Q})_0$ for all $t \in T$ and δ_t is the monodromy of δ_{t_0} along a path on T that connects t_0 to t. We will denote this space by $\mathrm{GHod}_n(X_{t_0}, \mathbb{Q})_0$.

On the other hand, as a consequence of the process of pole order reduction, we will observe that there are Hodge cycles of X that are independent of the polynomial P. Indeed, if $\{\delta_k\}_{k=0}^{m-2}$ are vanishing cycles, such that they are the basis for $H_0(\{P=1\})$ and $\{\delta_\alpha^{-1}\}_{\alpha\in J}$ are the basis for $H_{n-1}(\{g(x)=-1\})$ described in §2.3. We also know that $\{\delta_0 * \delta_\alpha^{-1},\dots,\delta_{m-2}*\delta_\alpha^{-1}\}_{\alpha\in J}$ is a basis for $H_n(U,\mathbb{Q})$. This means, each $\delta\in H_n(U,\mathbb{Q})$ is written

$$\delta=\sum_{k=0}^{m-2}\delta^k \text{ with } \delta^k=\sum_{\alpha\in J} n_{\alpha,k}\delta_k*\delta_\alpha^{-1},\qquad n_{\alpha,k}\in\mathbb{Q}, \tag{4.1}$$

where $J=I_{m_1}\times\cdots\times I_{m_n}$ with $I_{m_j}=\{0,1,2,\dots,m_j-2\}$. The condition of being Hodge cycle is given by the vanishing of the following integrals

$$\int_\delta res\left(\frac{\omega_\beta}{f^j}\right),\quad A_\beta<j\le\frac{n}{2}.$$

Using Proposition 2.3 and equality (4.1) we have

$$\int_\delta res\left(\frac{\omega_\beta}{f^j}\right)=\frac{1}{\zeta_q^{\gamma+1}-1}\sum_{k=0}^{m-2}\left(\sum_{\alpha\in J}n_{\alpha,k}\int_{\delta_\alpha^{-1}}\frac{\omega_{\beta'}}{dg}\right)\int_{\delta_k*_t\hat{\delta}} res\left(\frac{y^{\beta_{n+1}}z^\gamma dy\wedge dz}{(P_0-z^q)^j}\right),$$

with $\hat{\delta}=[\zeta_q-1]\in H_0(\{z^q=1\},\mathbb{Z})$, $A_{\beta'}=\frac{\gamma+1}{q}\notin\mathbb{N}$ and $\beta=(\beta',\beta_{n+1})$. Therefore if

$$\sum_{\alpha\in J}n_{k\alpha}\int_{\delta_\alpha^{-1}}\frac{\omega_{\beta'}}{dg}=0 \text{ for } k=0,\dots,m-2,$$

then

$$\int_\delta res\left(\frac{\omega_\beta}{f^j}\right)=0.$$

Observe that the above is also valid when $A_{\beta'}\in\mathbb{N}$ (see Proposition 2.3). Thus, we are tempted to define a subspace of the Hodge cycle space, and which we will call strong generic Hodge cycles as the image by the natural map of the following quotient

$$\frac{\left\{(n_{k\alpha})\in\mathbb{Q}^{|I|}\,\middle|\,\sum_{\alpha\in J}n_{k\alpha}\int_{\delta_\alpha^{-1}}\frac{\omega_{\beta'}}{dg}=0,\ \forall\beta \text{ s.t. } A_\beta<\frac{n}{2},k\in I_m\right\}}{\left\{(n_{k\alpha})\in\mathbb{Q}^{|I|}\,\middle|\,\sum_{\alpha\in J}n_{k\alpha}\int_{\delta_\alpha^{-1}}\frac{\omega_{\beta'}}{dg}=0,\ \forall\beta \text{ s.t. } A_\beta<n+1,k\in I_m\right\}}, \tag{4.2}$$

where $I=J\times\{0,1\dots,m-2\}$, $\beta=(\beta',\beta_{n+1})$ and $A_\beta=\sum_{j=1}^{n+1}\frac{\beta_j+1}{m_j}$.

Proposition 4.1. The following vector space is zero

$$\left\{\delta\in H_{n-1}(\{g=-1\},\mathbb{Q})\,\middle|\,\int_\delta\frac{\omega_{\beta'}}{dg}=0,\ \beta'\in J\right\},$$

where $J=I_{m_1}\times\cdots\times I_{m_n}$ with $I_{m_j}=\{0,1,2,\dots,m_j-2\}$. In particular the denominator of equation (4.2) is zero.

Proof. First, note that $\frac{\omega_{\beta'}}{dg} = \eta_{\beta'}$ (see equation (2.2)) and remember that the set of differential forms $\left\{\frac{\omega_{\beta'}}{dg} = \eta_{\beta'}\right\}_{\beta' \in J}$ are a basis for the $(n-1)$-th de Rham cohomology $H^{n-1}_{dR}(\{g=-1\})$ (see Proposition 2.5). On the other hand by Poincaré duality we have

$$H^{n-1}(\{g=-1\}, \mathbb{Q}) \cong H^{n-1}_c(\{g=-1\}, \mathbb{Q}) \cong H_{n-1}(\{g=-1\}, \mathbb{Q}).$$

This allows us to conclude the result (in the first isomorphism we use that the spaces are finite dimensional). For the second part just note that $A_\beta < n+1$ for $\beta = (\beta', 0)$ with $\beta' \in J$ and that $\delta \in H_{n-1}(\{g=-1\}, \mathbb{Q})$ is written

$$\delta = \sum_{\alpha \in J} n_\alpha \delta_\alpha^{-1}; \qquad n_\alpha \in \mathbb{Q}.$$

■

Thus, by the above proposition, using Proposition 2.6 and taking into account that

$$\left\{\beta' \middle| A_{\beta'} < \frac{n}{2} - \frac{1}{m}\right\} = \left\{\beta' \middle| A_\beta < \frac{n}{2},\ \beta = (\beta', \beta_{n+1})\right\},$$

we define

Definition 4.2. Consider the $\mathbb{Q}$-vector space

$$\begin{aligned}
\mathcal{A} :&= \left\{(n_{\alpha,k}) \in \mathbb{Q}^{|I|} \middle| \sum_{\alpha \in J} n_{\alpha,k} \int_{\delta_\alpha^{-1}} \frac{\omega_{\beta'}}{dg} = 0,\ \forall \beta \text{ s.t. } A_\beta < \frac{n}{2}\right\} \\
&= \left\{(n_{\alpha,k}) \in \mathbb{Q}^{|I|} \middle| \sum_{\alpha \in J} n_{\alpha,k} \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ \forall \beta \text{ s.t. } A_\beta < \frac{n}{2}\right\} \\
&= \left\{(n_{\alpha,k}) \in \mathbb{Q}^{|I|} \middle| \sum_{\alpha \in J} n_{\alpha,k} \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ \forall \beta' \text{ s.t. } A_{\beta'} < \frac{n}{2} - \frac{1}{m}\right\},
\end{aligned}$$

where $I = J \times\ I_m = I_{m_1} \times \cdots \times I_{m_n} \times I_m$, $\beta = (\beta', \beta_{n+1})$, and $A_\beta = \sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}$. The space of **strong generic Hodge cycles** is the image of $\mathcal{A}$ under the natural map

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathrm{Hod}_n(X, \mathbb{Q})_0 \\ (n_{\alpha,k}) & \longmapsto & \left[\sum_{k=0}^{m-2} \sum_{\alpha \in J} n_{\alpha,k} \delta_k * \delta_\alpha^{-1}\right]. \end{array} \tag{4.3}$$

We denote this space by $\mathrm{SHod}_n(X, \mathbb{Q})_0$.

Remark 4.1.

- We don't know if the natural map is injective.

- For $t_0 \in T$ and $t \in T$ in a neighborhood of t_0, the monodromy of $\delta_k^{t_0} * \delta_\alpha^{-1} \in H_n(U_{t_0}, \mathbb{Z})$ is given by $\delta_k^t * \delta_\alpha^{-1} \in H_n(U_t, \mathbb{Z})$ where δ_k^t is the monodromy of $\delta_k^{t_0}$ in the family

$$\mathcal{V} := \{(y,t) \in \mathbb{C} \times T |\ P_t(y) = 1\}.$$

  This implies that $\mathrm{SHod}_n(X_{t_0}, \mathbb{Q})_0 \subset \mathrm{GHod}_n(X_{t_0}, \mathbb{Q})_0$. We don't know when they are the same.

- If $g(x) = x_1^{m_1} + \cdots + x_n^{m_n} + x_{n+1}^2 + x_{n+2}^2 + \cdots + x_{n+k}^2$ with $m_j \geq 3$ then

$$\mathcal{A} \cong \left\{ (n_\alpha) \in \mathbb{Q}^{|J|} \left| \sum_{\alpha \in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ \sum_{j=1}^{n} \frac{\beta_j + 1}{m_j} < \frac{n}{2} - \frac{1}{m} \right. \right\}^{m-1},$$

  this means that the dimension of $\mathcal{A}$ depends on the number of variables with exponent greater than 2, and therefore the same is true for the rank of the strong generic Hodge cycles space.

Remark 4.2. It is easy observe that

$$\mathcal{A} \cong \left\{ (n_\alpha) \in \mathbb{Q}^{|J|} \left| \sum_{\alpha \in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ \forall \beta' \text{ s.t. } A_{\beta'} < \frac{n}{2} - \frac{1}{m} \right. \right\}^{m-1}, \tag{4.4}$$

and therefore $\delta = \sum_{k=0}^{m-2} \delta^k$ is a strong generic Hodge cycle if and only if $\{\delta^k\}_{k=0,\dots,m-2}$ are strong generic Hodge cycles, with $\delta^k = \sum_{\alpha \in J} n_{\alpha,k} \delta_k * \delta_\alpha^{-1}$.

Example 4.1. The vector space $\mathcal{A}$ is computable. For example for $g(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n^9$ and $P(y)$ is a polynomial of degree 3, we have that the vector space $\frac{\mathbb{Q}[x]}{\langle \Phi_9 \rangle}$ has a base $\{1, \zeta_9, \zeta_9^2, \dots, \zeta_9^5\}$, where $\Phi_9(z) = x^6 + x^3 + 1$ is the cyclotomic polynomial. By a straightforward calculation we get that

$$\mathcal{A} = \left\{ (n_1, n_2) \in \mathbb{Q}^8 \times \mathbb{Q}^8 \left| \begin{array}{l} n_{k0} = n_{k3} = n_{k6} \\ n_{k1} = n_{k4} = n_{k7} \quad \text{for } k = 0, 1 \\ \quad n_{k2} = n_{k5} = 0 \end{array} \right. \right\} \cong \mathbb{Q}^4. \tag{4.5}$$

Therefore, $\dim \mathrm{SHod}_n(X, \mathbb{Q}) \leq 4$. Analogously if $g(x) = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^3 + x_n^6$ and $P(y)$ is a polynomial of degree 3, we obtain

$$\mathcal{A} = \left\{ (n_{ij}) \in \mathbb{Q}^2 \times \mathbb{Q}^5 \left| \begin{array}{l} n_{00} - n_{03} + n_{12} = -n_{01} - n_{04} + n_{11} - n_{14} \\ n_{02} - n_{13} + n_{10} = -n_{01} - n_{04} + n_{11} - n_{14} \end{array} \right. \right\}^2 \cong \mathbb{Q}^{16},$$

Thus, $\dim \mathrm{SHod}_n(X, \mathbb{Q}) \leq 16$. See tables 4.1, 4.2, 4.3 and 4.4 for more dimensions of $\mathcal{A}$ in several cases.

In table 4.1 we can see some values for the dimension of $\mathcal{A}$ for the variety X induced by $g(x) = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_n} + x_n^{m_n}$ and P polynomial of degree m. It is believed that $\dim \mathcal{A} = (m-1)(m_n-1)$ for $m \geq 7$. This leads us to the next conjecture:

Conjecture 4.1. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x)+P(y)$, where $g(x) = x_1^2+\cdots+x_{n-2}^2+x_{n-1}^{m_n}+x_n^{m_n}$ and P is a polynomial of degree m. Then $\dim \mathrm{SHod}_n(X,\mathbb{Q}) \leq (m-1)(m_n-1)$.

The explicit calculations in Propositions 3.4, 3.5 and 3.6 allow us to obtain:

Proposition 4.2. Consider the family $\pi : \mathcal{U} \longrightarrow T$ with

$$\mathcal{U} := \{(x,y,t) \in \mathbb{C}^n \times \mathbb{C} \times T|\ f_t(x,y) := g(x) + P_t(y) = 0\},$$

and T the space of polynomials of degree 3 with nonzero discriminant. For $n = 2,4,6$ each generic Hodge cycle δ_{t_0} such that $\delta_{t_0} = \delta_{t_0}^k = \sum_{\alpha\in J} n_\alpha \delta_k^{t_0} * \delta_\alpha^{-1}$ in U_t for $k = 0,1$, is a strong generic Hodge cycle.

Proof. For each $t \in T$ we have

$$\int_{\delta_t} res\left(\frac{\omega_\beta}{(g+P_t)^j}\right) = 0, \quad A_\beta < j \leq \frac{n}{2},$$

where δ_t is the monodromy of δ_{t_0}. If $A_{\beta'} \notin \mathbb{N}$, then up to a nonzero number we have

$$\int_{\delta_t} res\left(\frac{\omega_\beta}{(g+P_t)^j}\right) = \left(\sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) \int_{\delta_k^t * \hat{\delta}} \frac{y^{\beta_{n+1}} z^\gamma dy\wedge dz}{(P_t - z^q)^j}.$$

(see Propositions 2.6, 2.3). As the integral of the right side is nonzero for $P = y(1-y)(\lambda-y)$ (see Propositions 3.4, 3.5, 3.6), we have

$$\sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0, \quad A_\beta < \frac{n}{2},\ A_{\beta'} \notin \mathbb{N}.$$

If $A_{\beta'} \in \mathbb{N}$, then up to a nonzero number we have

$$\int_{\delta_t} res\left(\frac{\omega_\beta}{(g+P_t)^j}\right) = \left(\sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right) \int_{\tilde{\delta}_k^t} res\left(\frac{y^{\beta_{n+1}} dy}{P_t^{j-A_{\beta'}}}\right),$$

see Propositions 2.6, 2.3. Observe that the integral of the right side is over a 0-dimensional cycle. For $P = y(1-y)(\lambda-y)$, a straightforward computation allows us to conclude that this integral is nonzero. Therefore

$$\sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0, \quad A_\beta < \frac{n}{2},\ A_{\beta'} \in \mathbb{N}.$$

We conclude that δ is strong generic Hodge cycle. ∎

Before continuing let us list some general results.

m \ m_n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	14	8	38	12	30	20	66	20	98	24	26	32	46	32	86	36	126
4	21	12	39	18	45	24	27	30	117	36	39	42	69	48	111	54	69
5	12	16	52	24	28	32	36	40	108	48	52	56	60	64	148	72	76
6	15	20	25	30	35	40	45	50	95	60	65	70	75	80	85	90	95
7	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114
8	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133
9	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152

Table 4.1: Dimension of $\mathcal{A}$ on the variety induced by $g(x) = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_n} + x_n^{m_n}$ and a polynomial $P(y)$ of degree m.

m \ m_n	8	9	10	12	14	15	16	18	20	21	22	24	25	26	27	28	30
3	6	4	10	14	2	12	6	10	14	4	2	14	8	2	4	6	18
4	9	6	3	15	3	6	9	9	9	6	3	15	0	3	6	9	9
5	4	8	4	12	4	8	4	12	4	8	4	12	0	4	8	4	12
6	5	10	5	5	5	0	5	5	5	0	5	5	0	5	0	5	5
7	6	0	6	6	6	0	6	6	6	0	6	6	0	6	0	6	6
10	9	0	9	9	9	0	9	9	9	0	9	9	0	9	0	9	9
15	14	0	14	14	14	0	14	14	14	0	14	14	0	14	0	14	14
20	19	0	19	19	19	0	19	19	19	0	19	19	0	19	0	19	19

Table 4.2: Dimension of $\mathcal{A}$ on the variety induced by $g(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ and a polynomial $P(y)$ of degree m.

m_{n-1} \ m_n	4	5	6	7	8	9	10	11	12	13	14	15
4	14	8	14	0	14	4	14	0	30	0	6	12
5	8	8	8	0	8	0	32	0	8	0	8	8
6	14	8	38	0	14	16	18	0	50	0	10	24
7	0	0	0	12	0	0	0	0	0	0	12	0
8	14	8	14	0	30	4	14	0	30	0	6	12
9	4	0	16	0	4	20	4	0	24	0	4	8
10	14	32	18	0	14	4	66	0	22	0	10	36
11	0	0	0	0	0	0	0	20	0	0	0	0
12	30	8	50	0	30	24	22	0	98	0	14	32
13	0	0	0	0	0	0	0	0	0	24	0	0
14	6	8	10	12	6	4	10	0	14	0	26	12
15	12	8	24	0	12	8	36	0	32	0	12	32

Table 4.3: Dimension of $\mathcal{A}$ on the variety induced by $g(x) = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_{n-1}} + x_n^{m_n}$ and a polynomial $P(y)$ of degree $m = 3$.

m_{n-1} \ m_n	4	5	6	7	8	9	10	11	12	13	14	15
4	27	0	9	0	27	0	9	0	27	0	9	0
5	0	36	0	0	0	0	36	0	0	0	0	36
6	9	0	45	0	9	18	9	0	45	0	9	18
7	0	0	0	54	0	0	0	0	0	0	54	0
8	27	0	9	0	63	0	9	0	27	0	9	0
9	0	0	18	0	0	72	0	0	18	0	0	18
10	9	36	9	0	9	0	81	0	9	0	9	36
11	0	0	0	0	0	0	0	90	0	0	0	0
12	27	0	45	0	27	18	9	0	99	0	9	18
13	0	0	0	0	0	0	0	0	0	108	0	0
14	9	0	9	54	9	0	9	0	9	0	117	0
15	0	36	18	0	0	18	36	0	18	0	0	126

Table 4.4: Dimension of $\mathcal{A}$ on the variety induced by $g(x) = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_{n-1}} + x_n^{m_n}$ and a polynomial $P(y)$ of degree $m = 10$.

Proposition 4.3 (Deligne's [Del82])**.** Let X be a smooth projective variety. If $\delta \in H_m(X, \mathbb{Q})$ is algebraic, then for every $\omega \in H^m_{dR}(X/k)$:

$$\frac{1}{(2\pi i)^{m/2}} \int_\delta \omega \in \overline{k},$$

where X/k denotes the variety over a field $k \subset \mathbb{C}$.

This naturally leads us to the following conjecture, which is a consequence of the Hodge's conjecture:

Conjecture 4.2. Let X be a smooth projective variety. If $\delta \in H_m(X; \mathbb{Q})$ is a Hodge cycle, then for every $\omega \in H^m_{dR}(X/k)$:

$$\frac{1}{(2\pi i)^{m/2}} \int_\delta \omega \in \overline{k},$$

where X/k denotes the variety over a field $k \subset \mathbb{C}$.

In the 2-dimensional case the Hodge's conjecture is true, namely

Theorem 4.1 (Lefschetz theorem on $(1,1)$-classes)**.** For X a smooth projective variety, every cohomology class $\omega \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ is algebraic. In fact $\omega = \eta_D$ for some divisor D on X and η_D is Poincaré dual of D.

Therefore the above theorem allows us to deduce

Corollary 4.1. Conjecture 4.2 is true for $m = 2$.

4.2 An upper bound

In this section, we will give an upper bound for the dimension of the space of strong generic Hodge cycles in certain cases.

Theorem 4.2. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$ and P is a polynomial of degree m.

i. For $m_1 = \cdots = m_{n-1} = 2$ and $m \geq 7$

$$\mathcal{A} \cong \begin{cases} \mathbb{Q}\,[\upsilon]^{m-1} & m_n \text{ even}, \\ 0 & m_n \text{ odd}, \end{cases}$$

with generator $\upsilon = (n_j^0)$ which satisfies $\prod_{\substack{e|m_n \\ 1\leq e<\frac{m_n}{2}}} \Phi_{\frac{m_n}{e}}(x) = \sum_{j=0}^{d-2} n_j^0 x^j$ where Φ_k is kth cyclotomic polynomial. Therefore

$$\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq \begin{cases} m-1 & m_n \text{ even}, \\ 0 & m_n \text{ odd}. \end{cases}$$

ii. For $m_1 = \cdots = m_{n-2} = 2$, m_{n-1} prime, $\gcd(m_{n-1}, m_n) = 1$ and $\frac{1}{m_{n-1}} + \frac{1}{m} < \frac{1}{2}$, we have $\mathrm{SHod}_n(X, \mathbb{Q})_0 = 0$.

iii. For m_j different prime numbers, we have $\mathrm{SHod}_n(X, \mathbb{Q})_0 = 0$.

Proof. For the first part consider

$$\mathcal{A}_{m_n,m} := \left\{ (n_j) \in \mathbb{Q}^{m_n - 1} \,\middle|\, \sum_{j=0}^{m_n-2} n_j \zeta_{m_n}^{j(\beta_n+1)} = 0,\ \forall \beta_n \text{ s.t. } \frac{\beta_n + 1}{m_n} < \frac{1}{2} - \frac{1}{m} \right\}.$$

Note that in this case, with $g = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$, we have $\mathcal{A} = \mathcal{A}_{m_n,m}^{m-1}$. Thus, it is enough to prove that

$$\mathcal{A}_{m_n,m} \cong \begin{cases} \mathbb{Q} & m_n \text{ even,} \\ 0 & m_n \text{ odd.} \end{cases}$$

For this, consider

$$S_{m_n,m} := \left\{ 1 \leq e < m_n \left(\frac{1}{2} - \frac{1}{m} \right) \,:\, e | m_n \right\} \tag{4.6}$$

and $Q_{(n_j)}(x) = \sum_{j=0}^{m_n-2} n_j x^j$. For each $(n_j) \in \mathcal{A}_{m_n,m}$ and $e \in S_{m_n,m}$ it is satisfied that $Q_{(n_j)}(\zeta^e) = 0$ because $\frac{e}{m_n} < \frac{1}{2} - \frac{1}{m}$. This means for each $(n_j) \in \mathcal{A}_{m_n,m}$ we have

$$\prod_{e \in S_{m_n,m}} \Phi_{m_n/e}(x) \,\Big|\, Q_{(n_j)}(x),$$

where Φ_k is kth cyclotomic polynomial. The above implies that $\mathcal{A}_{m_n,m} \cong \mathbb{Q}^{m_n - 1 - N_{m_n,m}}$ with $N_{m_n,m} := \sum_{e \in S_{m_n,m}} \varphi\left(\frac{m_n}{e}\right)$ and φ is the Euler's totient function, via the isomorphism

$$\begin{array}{ccc} \mathcal{A}_{m_n,m} & \xrightarrow{\cong} & \mathbb{Q}[x]_{m_n-2-N_{m_n,m}} \\ (n_j) & \longmapsto & \frac{Q_{(n_j)}(x)}{\prod_{e \in S_{m_n,m}} \Phi_{\frac{m_n}{e}}(x)}. \end{array}$$

On the other hand, note that for $m \geq 7$, we have that $S_{m_n,m} = \left\{ 1 \leq e < \frac{m_n}{2} \,:\, e | m_n \right\}$, and using that $\sum_{e | m_n} \varphi(m_n/e) = m_n$ we get

$$N_{m_n,m} = \begin{cases} m_n - 2 & m_n \text{ even,} \\ m_n - 1 & m_n \text{ odd.} \end{cases}$$

this allows us to conclude the proof in the first case.

The idea of the proof of the second case is similar in spirit to the first case. Consider

$$\mathcal{B}_{m_{n-1},m_n,m} := \left\{ (n_j) \in \mathbb{Q}^{m_n - 1} \,\middle|\, \sum_{j=0}^{m_n-2} n_j \zeta_{m_n}^{j(\beta_n+1)} = 0,\ \forall \beta_n \text{ s.t. } \frac{\beta_n + 1}{m_n} < 1 - \frac{1}{m} - \frac{1}{m_{n-1}} \right\},$$

$$\mathcal{A}_{m_{n-1},m_n,m} := \left\{ (n_{ij}) \in \mathbb{Q}^{(m_{n-1}-1)(m_n-1)} \;\middle|\; \begin{array}{l} \displaystyle\sum_{i=0}^{m_{n-1}-2}\sum_{j=0}^{m_n-2} n_{ij}\zeta_{m_{n-1}}^{i(\beta_{n-1}+1)}\zeta_{m_n}^{j(\beta_n+1)} = 0, \\ \forall(\beta_{n-1},\beta_n) \text{ s.t. } \dfrac{\beta_{n-1}+1}{m_{n-1}} + \dfrac{\beta_n+1}{m_n} < 1 - \dfrac{1}{m} \end{array} \right\}.$$

Note that in this case, with $g = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_{n-1}} + x_n^{m_n}$, we have $\mathcal{A} = \mathcal{A}^{m-1}_{m_{n-1},m_n,m}$. As $\gcd(m_{n-1}, m_n) = 1$ and m_{n-1} is a prime number, we have $[\mathbb{Q}(\zeta_{m_{n-1}}, \zeta_{m_n}) : \mathbb{Q}(\zeta_{m_n})] = m_{n-1} - 1$ and therefore the $\mathbb{Q}(\zeta_{m_n})$-vector space $\mathbb{Q}(\zeta_{m_{n-1}}, \zeta_{m_n})$ has basis $\{1, \zeta_{m_{n-1}}, \ldots, \zeta_{m_{n-1}}^{m_{n-1}-2}\}$. This implies that

$$\sum_{i=0}^{m_{n-1}-2}\sum_{j=0}^{m_n-2} n_{ij}\zeta_{m_{n-1}}^{i(\beta_{n-1}+1)}\zeta_{m_n}^{j(\beta_n+1)} = 0 \Longleftrightarrow \sum_{j=0}^{m_n-2} n_{ij}\zeta_{m_n}^{j(\beta_n+1)} = 0 \text{ for each } 0 \le i \le m_{n-1} - 2.$$

Therefore

$$\begin{aligned} \mathcal{A}_{m_{n-1},m_n,m} &\cong \left\{ (n_{ij}) \in \mathbb{Q}^{(m_{n-1}-1)(m_n-1)} \;\middle|\; \begin{array}{l} (n_{ij})_{j=0}^{m_n-2} \in \mathcal{B}_{m_{n-1},m_n,m}, \\ \text{for every } 0 \le i \le m_{n-1} - 2 \end{array} \right\} \\ &\cong (\mathcal{B}_{m_{n-1},m_n,m})^{m_{n-1}-1}. \end{aligned}$$

From the above, it is enough to prove that $\mathcal{B}_{m_{n-1},m_n,m} = 0$, when $\frac{1}{m_{n-1}} + \frac{1}{m} < \frac{1}{2}$. For this, consider

$$S_{m_{n-1},m_n,m} := \left\{ 1 \le e < m_n \left(1 - \frac{1}{m_{n-1}} - \frac{1}{m}\right);\ e|m_n \right\} \tag{4.7}$$

and $Q_{(n_j)}(x) = \sum_{j=0}^{m_n-2} n_j x^j$. Thus for each $(n_j) \in \mathcal{B}_{m_{n-1}.m_n,m}$ and $e \in S_{m_{n-1},m_n,m}$ it is satisfied that $Q_{(n_j)}(\zeta^e) = 0$ because $\frac{e}{m_n} < 1 - \frac{1}{m_{n-1}} - \frac{1}{m}$. This means that for each $(n_j) \in \mathcal{B}_{m_{n-1},m_n,m}$ we have

$$\prod_{e \in S_{m_{n-1}.m_n,m}} \Phi_{m_n/e}(x) \Big| Q_{(n_j)}(x),$$

where Φ_k is kth cyclotomic polynomial. The above implies that $\mathcal{B}_{m_{n-1},m_n,m} \cong \mathbb{Q}^{m_n-1-N_{m_{n-1},m_n,m}}$ with $N_{m_{n-1},m_n,m} := \sum_{e \in S_{m_{n-1},m_n,m}} \varphi\left(\frac{m_n}{e}\right)$ and φ is Euler's totient function, via isomorphism

$$\begin{array}{ccc} \mathcal{B}_{m_{n-1},m_n,m} & \xrightarrow{\cong} & \mathbb{Q}[x]_{d-2-N_{m_{n-1},m_n,m}} \\ (n_j) & \longmapsto & \dfrac{Q_{(n_j)}(x)}{\prod_{e \in S_{m_{n-1},m_n,m}} \Phi_{\frac{m_n}{e}}(x)}. \end{array}$$

The condition $\frac{1}{m_{n-1}} + \frac{1}{m_n} < \frac{1}{2}$ guarantees that $S_{m_{n-1},m_n,m} = \left\{1 \le e \le \frac{m_n}{2};\ e|m_n\right\}$, and using that $\sum_{e|d} \varphi(d/e) = d$ we get $N_{m_{n-1},m_n,m} = m_n - 1$. Thus $\mathcal{B}_{m_{n-1},m_n,m} = 0$.

It remains for us to prove the third case. For this consider

$$\mathcal{A}_{(m_j)} \cong \left\{ (n_\alpha) \in \mathbb{Q}^{|J|} \,\middle|\, \sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ \forall \beta' \text{ s.t. } A_{\beta'} < \frac{n}{2} - \frac{1}{m} \right\},$$

where $J = \prod_{j=1}^n I_{m_j}$, $I_{m_j} = \{0, 1, \ldots, m_j - 2\}$ and $A_{\beta'} = \sum_{j=1}^n \frac{\beta_j+1}{m_j}$. Observe that $\mathcal{A} = \mathcal{A}_{(m_j)}^{m-1}$. It is sufficient to show that $\mathcal{A}_{(m_j)} = 0$. As $m_1, \ldots, m_n$ are different primes, we have

$$\mathbb{Q}(\zeta_{m_1}, \ldots, \zeta_{m_n}) \cong \mathbb{Q}(\zeta_{\prod m_j}) \text{ and } [\mathbb{Q}(\zeta_{\prod m_j}) : \mathbb{Q}] = \prod_{j=1}^{n} (m_j - 1).$$

Moreover note that if $\alpha \neq \hat{\alpha}$, then $\prod_{j=1}^n \zeta_{m_j}^{\alpha_j} \neq \prod_{j=1}^n \zeta_{m_j}^{\hat{\alpha}_j}$. Therefore the $\mathbb{Q}$-vector space $\mathbb{Q}(\zeta_{m_1}, \ldots, \zeta_{m_n})$ has basis $\left\{ \prod_{j=1}^n \zeta_{m_j}^{\alpha_j} \right\}_{\alpha\in J}$. This implies that

$$\sum_{\alpha\in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j} = 0 \Longleftrightarrow n_\alpha = 0 \ \text{ for every } \alpha \in J,$$

but the above is one of the conditions that satisfy the elements of $\mathcal{A}_{(m_j)}$, with $\beta' = (0, \ldots, 0)$. ■

The proof of Theorem 4.2 also provides a method to calculate the dimension of $\mathcal{A}_{m_n,m}$ when $m < 7$ and the dimension of $\mathcal{B}_{m_{n-1},m_n,m}$ when $\frac{1}{m_{n-1}} + \frac{1}{m} \geq \frac{1}{2}$. With this, we obtain:

Corollary 4.2. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$ and P is a polynomial of degree m.

i. For $m_1 = \cdots = m_{n-1} = 2$ and $m = 2, \ldots, 6$

$$\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq (m-1) \left(\sum_{\substack{2 \leq d \leq \lfloor \frac{2m}{m-2} \rfloor \\ d|m_n}} \varphi(d) \right),$$

where φ is the Euler's totient function. When $m = 2$ means that $2 \leq d \leq m_n$ and $d|m_n$. Therefore for $m = 2$, $\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq (m_n - 1)$.

ii. For $m_1 = \cdots = m_{n-2} = 2$, m_{n-1} prime, $\gcd(m_{n-1}, m_n) = 1$ and $\frac{1}{m_{n-1}} + \frac{1}{m} \geq \frac{1}{2}$, we have

$$\dim \mathrm{SHod}_n(X, \mathbb{Q})_0 \leq (m-1)(m_{n-1} - 1) \left(\sum_{\substack{2 \leq d \leq \lfloor \frac{m m_{n-1}}{m m_{n-1} - m - m_{n-1}} \rfloor \\ d|m_n}} \varphi(d) \right),$$

where φ is the Euler's totient function.

Proof. With the notations of the proof of Theorem 4.2, we have that $\mathcal{A} = \mathcal{A}_{m_n,m}^{m-1}$ and $\mathcal{A}_{m_n,m} \cong \mathbb{Q}^{m_n-1-N_{m_n,m}}$ with $N_{m_n,m} := \sum_{e\in S_{m_n,m}} \varphi\left(\frac{m_n}{e}\right)$, where $S_{m_n,n}$ is defined in (4.6). Observe that

$$N_{m_n,m} = m_n - \sum_{\substack{e\notin S_{m_n,m}\\ e|m_n}} \varphi\left(\frac{m_n}{e}\right).$$

Therefore

$$\begin{aligned}
\dim(\mathcal{A}) &= (m-1)\left(\sum_{\substack{e\notin S_{m_n,m}\\ e|m_n}} \varphi\left(\frac{m_n}{e}\right) - 1\right)\\
&= (m-1)\left(\sum_{\substack{m_n(\frac{1}{2}-\frac{1}{m})\leq e<m_n\\ e|m_n}} \varphi\left(\frac{m_n}{e}\right)\right)\\
&= (m-1)\left(\sum_{\substack{2\leq d\leq \frac{2m}{m-2}\\ d|m_n}} \varphi\left(d\right)\right),
\end{aligned}$$

where $d = \frac{m_n}{e}$. This proves the first part. For the second part, we know from the proof of Theorem 4.2 that $\mathcal{A} \cong \mathcal{B}_{m_{n-1},m_n,m}^{(m-1)(m_{n-1}-1)}$ and $\mathcal{B}_{m_{n-1},m_n,m} \cong \mathbb{Q}^{m_n-1-N_{m_{n-1},m_n,m}}$ with $N_{m_{n-1},m_n,m} := \sum_{e\in S_{m_{n-1},m_n,m}} \varphi\left(\frac{m_n}{e}\right)$, where $S_{m_{n-1},m_n,m}$ is defined in (4.7). With this, we proceed as in the first part. ∎

Remark 4.3. The bound of Corollary 4.2 depends on m_n modulo some integer that depends on m and m_{n-1}. For example: let X be a desingularization of the weighted hypersurface D given by quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ and $P(y)$ is a polynomial of degree 4. We have

$$\dim \mathrm{SHod}_n(X,\mathbb{Q})_0 \leq \begin{cases} 0 & m_n \equiv 1,5,7,11\ (12),\\ 3 & m_n \equiv 2,10\ (12),\\ 6 & m_n \equiv 3,9\ (12),\\ 9 & m_n \equiv 4,6,8\ (12),\\ 15 & m_n \equiv 0\ (12).\end{cases}$$

One more example: let X be a desingularization of the weighted hypersurface D given by quasi-homogenization F of $f = g(x)+P(y)$, where $g(x) = x_1^2+x_2^2+\cdots+x_{n-2}^2+x_{n-1}^{m_{n-1}}+x_n^{m_n}$, $(m_{n-1}, m_n) = 1$ and $P(y)$ is a polynomial of degree 3. We have

$$\dim \mathrm{SHod}_n(X,\mathbb{Q})_0 \leq \begin{cases} 0 & m_n \equiv 1,5\ (6)\\ 4 & m_n \equiv 2,4\ (6)\end{cases} \quad \text{for } m_{n-1} = 3,$$

$$\dim \mathrm{SHod}_n(X,\mathbb{Q})_0 \leq \begin{cases} 0 & m_n \text{ odd} \\ 8 & m_n \text{ even} \end{cases} \quad \text{for } m_{n-1}=5.$$

A small reflection on the proof of the third part of Theorem 4.2 allows us to deduce

Corollary 4.3. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$ and $P(y) = y^{m_{n+1}} + 1$. If m_j, $j = 1, \ldots, n+1$, are different prime numbers, then $\mathrm{Hod}_n(X,\mathbb{Q})_0 = 0$.

Proof. Let us consider $t_0 = (1, 0, \ldots, 0, 1) \in T$. In this case $f_{t_0} = x_1^{m_1} + \cdots + x_n^{m_n} + y^{m_{n+1}} + 1$, and we can write equation (3.3) as

$$\left[\frac{\omega_\beta}{f^j}\right] = C_j \left[\frac{\omega_\beta}{f}\right],$$

with $C_j \in \mathbb{Q}$. If $A_\beta \in \mathbb{N}$ and $A_\beta < j$, then $C_j = 0$. With this and using Proposition 2.6 and 2.4, we have in the definition of Hodge cycles (see Definition 1.8)

$$\left\{\delta \in H_n(U_{t_0},\mathbb{Q}) \Big| \int_\delta res\left(\frac{\omega_\beta}{f_{t_0}^j}\right) = 0, A_\beta < j, 1 \leq j \leq \frac{n}{2}\right\} \cong \mathcal{A}_{(m_j)},$$

where

$$\mathcal{A}_{(m_j)} \cong \left\{(n_\alpha) \in \mathbb{Q}^{|I|} \left| \sum_{\alpha\in I} n_\alpha \prod_{j=1}^{n+1} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0, \ \forall\beta \text{ s.t. } A_\beta < \frac{n}{2}, A_\beta \notin \mathbb{N} \right.\right\},$$

where $I = \prod_{j=1}^{n+1} I_{m_j}$, $I_{m_j} = \{0, 1, \ldots, m_j - 2\}$ and $A_\beta = \sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}$. Similarly, as in the third part of the proof of Theorem 4.2, we have $\mathcal{A}_{(m_j)} = 0$, when $m_1, \ldots, m_{n+1}$ are different prime numbers. In conclusion $\mathrm{Hod}_n(X_{t_0},\mathbb{Q})_0 = 0$, where X_{t_0} is a desingularization of the weighted hypersurface D_{t_0} given by the quasi-homogenization F_{t_0} of f_{t_0} and m_j, $j = 1, \ldots, n+1$ are different prime numbers. ■

4.3 Hodge numbers

With the notations from the previous sections, let us consider $t_0 = (1, 0, \ldots, 0, 1) \in T$. In this case $f_{t_0} = x_1^{m_1} + \cdots + x_n^{m_n} + y^{m_{n+1}} + 1$. The affine Fermat variety $\{f_{t_0} = 0\}$ has a sequence of numbers related to the Hodge numbers of the compact smooth underlying variety X_{t_0}. Namely

$$h_0^{k-1,n-k+1} := \#\{\beta \in I |\ k-1 < A_\beta < k\}.$$

These number are symmetric, $h_0^{k-1,n-k+1} = h_0^{n-k+1,k-1}$, since the set I is invariant under the transformation

$$\beta \longrightarrow\ m - \beta - 2 := (m_1 - \beta_1 - 2, \ldots, m_{n+1} - \beta_{n+1} - 2),$$

and therefore $A_{m-\beta-2} = n + 1 - A_\beta$. These numbers satisfy $h^{p,q} = h_0^{p,q}$, for $p \neq q$, where $h^{p,q} = \dim H^{p,q}(X)$. In the remaining case $h^{\frac{n}{2},\frac{n}{2}} - h_0^{\frac{n}{2},\frac{n}{2}}$ depends on the desingularization of D_{t_0}. This can be deduced from Example 1.1. For more details see [Mov20, §15.4]. The Hodge numbers do not change when the complex structure is varied continuously. More precisely

Theorem 4.3. Let $\pi : \mathcal{X} \longrightarrow B$ be a family of complex manifolds (i.e. π is proper and submersive) and assume that $\mathcal{X}_0$ is Kähler for some $0 \in B$. Then for b in a neighborhood of 0 in B, the Hodge numbers of $\mathcal{X}_b$ are the same as the Hodge numbers of $\mathcal{X}_0$.

Proof. See [Voi02, Proposition 9.20]. ∎

The above theorem implies that the numbers $h_0^{k-1,n-k+1}$ are the same for every $t \in T$ in the fibration $\pi : \mathcal{U} \longrightarrow T$.

Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n} + x_{n+1}^2 + x_{n+2}^2 + \cdots + x_{n+k_1}^2$ and $P(y)$ is a fixed polynomial of degree m. Observe that $\frac{k_1}{2} < A_\beta$ and this implies that $h_0^{k-1,n-k+1} = 0 = h_0^{n-k+1,k-1}$ for $k \leq \lfloor \frac{k_1}{2} \rfloor$. For example if $g = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ or $g = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_{n-1}} + x_n^{m_n}$, the sequence of numbers are surface-like:

$$0, \ldots, 0, h_0^{\frac{n}{2}+1, \frac{n}{2}-1}, h_0^{\frac{n}{2}, \frac{n}{2}}, h_0^{\frac{n}{2}-1, \frac{n}{2}+1}, 0, \ldots, 0.$$

In other words, the Hodge structure of $H^n(X, \mathbb{Z})$ has level 2. Further, if $\mathbf{h}_0^{2,0}, \mathbf{h}_0^{1,1}$ are the corresponding Hodge numbers for $g_2 = x_1^2 + x_2^{m_n}$ or $g_2 = x_1^{m_{n-1}} + x_2^{m_n}$ respectively, we have $\mathbf{h}_0^{2,0} = h_0^{\frac{n}{2}+1, \frac{n}{2}-1}$ and $\mathbf{h}_0^{1,1} = h_0^{\frac{n}{2}, \frac{n}{2}}$. In some cases we can calculate the Hodge numbers. For example

Corollary 4.4. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ and $P(y)$ is a polynomial of degree m_n. We have

$$h_0^{\frac{n}{2}+1, \frac{n}{2}-1} = h_0^{\frac{n}{2}-1, \frac{n}{2}+1} = \begin{cases} \frac{m_n(m_n-6)}{8} + 1 & m_n \text{ even}, \\ \frac{(m_n+1)(m_n-5)}{8} + 1 & m_n \text{ odd}, \end{cases}$$

$$h_0^{\frac{n}{2}, \frac{n}{2}} = \begin{cases} \frac{3m_n^2 - 6m_n + 4}{4} & m_n \text{ even}, \\ \frac{(m_n-1)(3m_n-1)}{4} & m_n \text{ odd}. \end{cases}$$

Proof. First, note that it is enough to prove it for the case $n = 2$. Let us see one case, the others are analogous. Suppose m_n even, thus

$$
\begin{aligned}
h_0^{1,1} :=& \#\{\beta \in I|\ 1 < A_\beta < 2\} \\
=& \# \left\{ \beta \in I \middle| \ \frac{m_n}{2} - 2 < \beta_2 + \beta_3 < \frac{3m_n}{2} - 2 \right\} \\
=& \sum_{j=\frac{m_n}{2}-1}^{m_n-2} (j+1) + \sum_{j=m_n-1}^{\frac{3m_n}{2}-3} (2m_n - 3 - j) \\
=& \sum_{j=\frac{m_n}{2}-1}^{m_n-2} (j+1) + \sum_{j=\frac{m_n}{2}}^{m_n-2} j \\
=& \frac{3m_n^2 - 6m_n + 4}{4}.
\end{aligned}
$$

■

The variety induced by $f = g(x) + P(y)$, where $g = x_1^2 + \cdots + x_{n-1}^2 + x_n^d$ and $P(y) = y(1-y)(\lambda - y)$, will be used constantly in the next section. In the following corollary, we calculate their Hodge numbers.

Corollary 4.5. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ and $P(y)$ is a polynomial of degree 3. We have

$$
h_0^{\frac{n}{2}+1, \frac{n}{2}-1} = h_0^{\frac{n}{2}-1, \frac{n}{2}+1} = \left\lfloor \frac{m_n - 1}{6} \right\rfloor,
$$

$$
h_0^{\frac{n}{2}, \frac{n}{2}} = m_n - 2 + \left\lceil \frac{5m_n}{6} \right\rceil - \left\lfloor \frac{m_n}{6} \right\rfloor.
$$

Proof. The proof is similar to the previous corollary. ■

m \ m_n	8	9	10	11	13	14	17	19	20
3	(1,12)	(1,14)	(1,16)	(1,18)	(2,20)	(2,22)	(2, 28)	(3,30)	(3,32)
4	(1,17)	(2, 20)	(2 23)	(2,27)	(3,30)	(3, 33)	(4,40)	(4, 46)	(4,47)
5	(2,24)	(2,28)	(2,28)	(4,36)	(4,40)	(5,42)	(6,52)	(6,60)	(6,60)
10	(6,51)	(6,60)	(6,61)	(10,79)	(11,86)	(12,93)	(15,114)	(16,130)	(16,131)

Table 4.5: Hodge numbers $\left(h^{\frac{n}{2}+1,\frac{n}{2}-1}, h_0^{\frac{n}{2},\frac{n}{2}}\right)$ of the variety induced by $f = x_1^2 + \cdots + x_{n-1}^2 + x_n^{m_n} + P(y)$, with $P(y)$ is a polynomial of degree m.

p \ d	8	9	10	12	13	14
3	(2,24)	(2,26)	(3,30)	(3,36)	(3,40)	(4,44)
4	(4,34)	(4, 40)	(5,44)	(5,50)	(8,56)	(8, 62)
7	(9,66)	(10,76)	(11,86)	(15,102)	(16,112)	(18,120)
8	(11,76)	(11,90)	(14,98)	(16,116)	(20,128)	(21,140)
12	(16,116)	(20,134)	(22,152)	(24,174)	(34,196)	(35,214)

Table 4.6: Hodge numbers $\left(h^{\frac{n}{2}+1,\frac{n}{2}-1}, h_0^{\frac{n}{2},\frac{n}{2}}\right)$ of the variety induced by $f = x_1^2 + \cdots + x_{n-2}^2 + x_{n-1}^{m_{n-1}} + x_n^{m_n} + P(y)$ with $P(y)$ is a polynomial of degree 3.

4.4 Strong generic Hodge cycles and hypergeometric function

In the rest of the document we will use d instead of m_n. The proof of Theorem 4.2 provides us with a method to find strong generic Hodge cycles explicitly in the cases described. In the 2-dimensional case, by Lefschetz $(1,1)$ theorem each Hodge cycle is algebraic. But the algebraic cycles satisfy the property of Proposition 4.3. Thus, we can find algebraic expressions involving hypergeometric functions using Proposition 4.3 and the following fact

Lemma 4.1.

$$\frac{B\left(\frac{1}{2},\frac{\beta_2+1}{d}\right)B\left(\frac{1}{2}+\frac{\beta_2+1}{d}+k,\frac{1}{2}+\frac{\beta_2+1}{d}\right)}{\pi}\in\overline{\mathbb{Q}},$$

with $k\in\mathbb{Z}$ and $\beta_2\in\mathbb{N}$. Additionally

$$\frac{B\left(\frac{1}{2},\ldots,\frac{1}{2},\frac{\beta_n+1}{d}\right)B\left(A_{\beta'}+k,A_{\beta'}\right)}{\pi^{\frac{n}{2}}} = Q\frac{B\left(\frac{1}{2},\frac{\beta_n+1}{d}\right)B\left(\frac{1}{2}+\frac{\beta_n+1}{d}+k,\frac{1}{2}+\frac{\beta_n+1}{d}\right)}{\pi},\quad Q\in\mathbb{Q},$$

where $A_{\beta'} = \frac{n-1}{2}+\frac{\beta_n+1}{d}$. In particular, the last expression is algebraic.

Proof. It is enough to prove it for $k=0$ since $B(a+1,b)=\frac{a}{a+b}B(a,b)$.

$$\begin{aligned}
B\left(\frac{1}{2},\frac{\beta_2+1}{d}\right)B\left(\frac{1}{2}+\frac{\beta_2+1}{d},\frac{1}{2}+\frac{\beta_2+1}{d}\right) &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\beta_2+1}{d}\right)\Gamma\left(\frac{1}{2}+\frac{\beta_2+1}{d}\right)}{\Gamma\left(1+\frac{2(\beta_2+1)}{d}\right)}\\
&= \frac{2^{1-\frac{2(\beta_2+1)}{d}}\pi\Gamma\left(\frac{2(\beta_2+1)}{d}\right)}{\Gamma\left(1+\frac{2(\beta_2+1)}{d}\right)}\\
&= \frac{2^{\frac{-2(\beta_2+1)}{d}}\pi d}{\beta_2+1}.
\end{aligned}$$

For the second part consider

$$\mathcal{D} = B\left(\frac{1}{2}, \frac{\beta_n+1}{d}\right) B\left(\frac{1}{2}+\frac{\beta_n+1}{d}, \frac{1}{2}+\frac{\beta_n+1}{d}\right),$$

therefore

$$\begin{aligned}
B\left(\frac{1}{2},\ldots,\frac{1}{2},\frac{\beta_n+1}{d}\right) B\left(A_{\beta'},A_{\beta'}\right) =& \frac{\Gamma\left(\frac{1}{2}\right)^{n-1}\Gamma\left(\frac{\beta_n+1}{d}\right)\Gamma\left(A_{\beta'}\right)}{\Gamma(2A_{\beta'})} \\
=& \frac{\Gamma\left(\frac{1}{2}\right)^{n-2}\Gamma\left(1+\frac{2(\beta_n+1)}{d}\right)\Gamma\left(\frac{n-1}{2}+\frac{\beta_n+1}{d}\right)}{\Gamma\left(\frac{1}{2}+\frac{\beta_n+1}{d}\right)\Gamma\left(n-1+\frac{2(\beta_n+1)}{d}\right)}\mathcal{D} \\
=& \frac{\prod_{j=0}^{\frac{n}{2}-2}\left(\frac{1}{2}+\frac{\beta_n+1}{d}+j\right)}{\prod_{j=1}^{n-2}\left(\frac{2(\beta_n+1)}{d}+j\right)}\pi^{\frac{n}{2}-1}\mathcal{D}.
\end{aligned}$$

In the above we have used that $\Gamma(z+k)=\prod_{j=0}^{k-1}(z+j)\Gamma(z)$. This implies that

$$\frac{B\left(\frac{1}{2},\ldots,\frac{1}{2},\frac{\beta_n+1}{d}\right)B\left(A_{\beta'},A_{\beta'}\right)}{\pi^{\frac{n}{2}}} \in \overline{\mathbb{Q}}.$$

■

In most of the following results, we will have the hypothesis that $A_{\beta'} \notin \mathbb{N}$. This assures that hypergeometric functions appear within the computations of the periods (see Propositions 2.2, 3.4, 3.5). In the 2-dimensional case every Hodge cycle is an algebraic cycle by Lefschetz $(1,1)$ theorem. Thus, we can deduce the following result

Proposition 4.4. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x)+P(y)$, where $g(x) = x_1^2 + x_2^d$ and $P(y) = y(1-y)(\lambda-y)$. Consider $\frac{\omega_\beta}{f}$ a good form (see Definition 3.2) with $A_{\beta'} = \frac{1}{2}+\frac{\beta_2+1}{d} \notin \mathbb{N}$ and $\beta = (\beta', \beta_3) = (\beta_1,\beta_2,\beta_3)$. Let

$$\delta^0 = \sum_{j=0}^{d-2} n_{j,0}\delta_0 * \delta_j^{-1}, \quad \delta^1 = \sum_{j=0}^{d-2} n_{j,1}\delta_1 * \delta_j^{-1}.$$

If δ^0 and δ^1 are generic Hodge cycles then either

$$\sum_{j=0}^{d-2} n_{j,k}\zeta_d^{j(\beta_2+1)}, \quad k=0,1 \tag{4.8}$$

is zero or

$$F\left(A_{\beta'}+\beta_3, 1-A_{\beta'}, 2A_{\beta'}+\beta_3; \frac{1}{\lambda}\right) \text{ and } F\left(A_{\beta'}, 1-A_{\beta'}-\beta_3, 2A_{\beta'}; 1-\lambda\right)$$

are in $\overline{\mathbb{Q}(\lambda)}$.

Proof. The fact that $\frac{\omega_\beta}{f}$ is a good form means that $res\left(\frac{\omega_\beta}{f}\right) \in H^2_{dR}(X/\mathbb{Q})$ (see Proposition 3.7). Therefore by Lefschetz (1.1) theorem and Proposition 4.3 we have

$$\frac{1}{2\pi i}\int_\delta res\left(\frac{\omega_\beta}{f}\right) \in \overline{\mathbb{Q}(\lambda)},$$

if δ is a generic Hodge cycle. The above integral is computed in Proposition 3.4 and using Lemma 4.1 the result is obtained. ■

Now, as an application we will re-obtain results already shown by Schwarz in [Sch73] (See [Beu10, BH89] for some generalizations.)

Corollary 4.6. The following expressions are in $\overline{\mathbb{Q}(\lambda)}$

$$F\left(\frac{5}{6},\frac{1}{6}-\beta_3,\frac{5}{3};1-\lambda\right),\quad F\left(\frac{7}{6},\frac{-1}{6}-\beta_3,\frac{7}{3};1-\lambda\right),$$

$$F\left(\frac{5}{6}+\beta_3,\frac{1}{6},\frac{5}{3}+\beta_3;\frac{1}{\lambda}\right),\quad F\left(\frac{7}{6}+\beta_3,\frac{-1}{6},\frac{7}{3}+\beta_3;\frac{1}{\lambda}\right),$$

with $\beta_3 = 0, 1$.

Proof. We will give two proofs: The first one using the theory developed so far and the second one using classical theory of hypergeometric functions. For the first proof let X be a desingularization of the weighted hypersurface D given by quasi-homogenization F of $f = g(x)+P(y)$, where $g(x) = x_1^2+x_2^9$ and $P(y) : y(1-y)(\lambda-y)$. Consider

$$\delta^1 = n_0(\delta_1 * \delta_0^{-1} + \delta_1 * \delta_3^{-1} + \delta_1 * \delta_6^{-1}) + n_1(\delta_1 * \delta_1^{-1} + \delta_1 * \delta_4^{-1} + \delta_1 * \delta_7^{-1}),$$

which is a strong generic Hodge cycle (see equation (4.5). Now, take $\beta = (0, \beta_2, \beta_3)$ with $\beta_2 = 2, 5$. In this case the differential form $\frac{\omega_\beta}{f}$ is a good form (see Example 3.2) and observe that

$$n_0\left(\zeta_9^{0(\beta_2+1)} + \zeta_9^{3(\beta_2+1)} + \zeta_9^{6(\beta_2+1)}\right) + n_1\left(\zeta_9^{1(\beta_2+1)} + \zeta_9^{4(\beta_2+1)} + \zeta_9^{7(\beta_2+1)}\right) \neq 0.$$

Therefore, using the previous proposition we obtain the first two expressions of the corollary. To obtain the last two, we consider the same β's and the following strong generic Hodge cycle

$$\delta^0 = n_0(\delta_0 * \delta_0^{-1} + \delta_0 * \delta_3^{-1} + \delta_0 * \delta_6^{-1}) + n_1(\delta_0 * \delta_1^{-1} + \delta_0 * \delta_4^{-1} + \delta_0 * \delta_7^{-1}).$$

Let us continue with the second proof. The hypergeometric function $F(5/6, 1/6, 5/3; z)$ is contiguous (see Definition A.1) to $F(-1/6, 1/6, 2/3; z)$ which has angular parameters $(1/3, 2/3, 1/3)$ and this triplet is in Table A.1. Therefore applying Corollary A.1 we have the first and the third equations of corollary are algebraic. The same argument is valid for the other two equations, observing that $F(7/6, -1/6, 7/3; z)$ is contiguous to $F(1/6, -1/6, 1/3; z)$ which has angular parameters $(2/3, 1/3, 1/3)$. ■

The property in Proposition 4.3 would be also true for Hodge cycles if the Hodge conjecture is true. Deligne has proved this property for Hodge cycles in the usual Fermat variety, even though the Hodge conjecture is unknown. More explicitly

Proposition 4.5 (Deligne's [Del82])**.** Let X be a smooth projective variety defined by $x_1^d+\cdots+x_{n+1}^d$. If $\delta \in H_m(X,\mathbb{Q})$ is a Hodge cycle, then for every $\omega \in H^m_{dR}(X/k)$:

$$\frac{1}{(2\pi i)^{m/2}}\int_\delta \omega \in \overline{k},$$

where X/k denotes the variety over a field $k\subset\mathbb{C}$.

In this same direction, we have the following result

Proposition 4.6. Let X_n be a desingularization of the weighted hypersurface D_n given by the quasi-homogenization F_n of $f_n = g_n(x)+P(y)$, where $g_n(x) = x_1^2+\cdots+x_{n-1}^2+x_n^d$ and $P(y) = y(1-y)(\lambda-y)$. Consider $\frac{\omega_\beta}{f_n}$ a good form (see Definition 3.2) with $A_{\beta'}\notin\mathbb{N}$. If $\delta\in H_n(X_n,\mathbb{Q})$ is a strong generic Hodge cycle, we have

$$\frac{1}{(2\pi i)^{n/2}}\int_\delta res\left(\frac{\omega_\beta}{f_n}\right)\in\overline{\mathbb{Q}(\lambda)},$$

Proof. The main idea of the proof is to use Lefschetz $(1,1)$ theorem in the 2-dimensional case and to construct the n-dimensional integral from the 2-dimensional integral. Consider $(n_{kj})\in\mathcal{A}$, where $\mathcal{A}$ is defined in (4.4). This element induces the cycle

$$\delta^n = \delta^{n0}+\delta^{n1} = \sum_{j=0}^{d-2} n_{j,0}\delta_0 * \delta_{n,j}^{-1} + \sum_{j=0}^{d-2} n_{j,1}\delta_1 * \delta_{n,j}^{-1},$$

with $\delta_{n,j}^{-1}\in H_{n-1}(\{g_n=-1\})$, which in turn induces a strong generic Hodge cycle. We know that δ^n is a strong generic Hodge cycle if and only if δ^{n0},δ^{n1} are strong generic Hodge cycles (see Remark 4.2). Now consider the differential form $\frac{\omega_{\hat\beta}}{f_2}$ with $\hat\beta=(0,\beta_n,\beta_{n+1})$. Observe that $\frac{\omega_{\hat\beta}}{f_2}=\frac{\omega_\beta}{f_n}$ for $n=2$ and that $\frac{n}{2}-1<A_\beta=A_{\hat\beta}+\frac{n}{2}-1$ for each β. An analysis similar to the proof of Proposition 4.10 allows us to deduce that if $\frac{\omega_\beta}{f_n}$ is a good form then $\frac{\omega_{\hat\beta}}{f_2}$ is a good form. Now, by Proposition 3.4, up to multiplication by a nonzero element of $\overline{\mathbb{Q}(\lambda)}$ we have

$$\begin{aligned}\frac{1}{(2\pi i)^{\frac{n}{2}}}\int_{\delta^{n0}} res\left(\frac{\omega_\beta}{f_n}\right) = \frac{1}{(2\pi i)^{\frac{n}{2}}}\left(\sum_{j=0}^{d-2} n_{j,0}\zeta_d^{j(\beta_n+1)}\right) B\left(\frac{1}{2},\ldots,\frac{1}{2},\frac{\beta_n+1}{d}\right)\times \\ B\left(A_{\beta'},A_{\beta'}\right)F\left(A_{\beta'}+\beta_{n+1},1-A_{\beta'},2A_{\beta'}+\beta_{n+1};\frac{1}{\lambda}\right).\end{aligned} \tag{4.9}$$

On the other hand, up to multiplication by a nonzero element of $\overline{\mathbb{Q}(\lambda)}$

$$\begin{aligned}\frac{1}{2\pi i}\int_{\delta^{20}} res\left(\frac{\omega_{\hat\beta}}{f_2}\right) = \frac{1}{2\pi i}\left(\sum_{j=0}^{d-2} n_{j,0}\zeta_d^{j(\beta_n+1)}\right) B\left(\frac{1}{2},\frac{\beta_n+1}{d}\right)\times \\ B\left(A_{\hat\beta'},A_{\hat\beta'}\right)F\left(A_{\hat\beta'}+\beta_{n+1},1-A_{\hat\beta'},2A_{\hat\beta'}+\beta_{n+1};\frac{1}{\lambda}\right).\end{aligned} \tag{4.10}$$

Since δ^{20} induces a strong generic Hodge cycle of X_2, by Lefschetz $(1,1)$ theorem, δ^{20} is algebraic. By Proposition 4.3 we have that $\frac{1}{2\pi i}\int_{\delta^{20}} res\left(\frac{\omega_{\hat{\beta}}}{f_2}\right) \in \overline{\mathbb{Q}(\lambda)}$. If the integral in (4.10) is zero, then equation (4.9) is zero. If it is not zero, then Lemma 4.1 allows us to conclude that the hypergeometric function of equation (4.10) is algebraic over $\mathbb{Q}(\lambda)$. As $A_{\beta'} = A_{\hat{\beta}'} + \frac{n}{2} - 1$ and $\frac{\omega_{\hat{\beta}}}{f_2}$ is a good form we have that the functions

$$F\left(A_{\beta'} + \beta_{n+1}, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1}; \frac{1}{\lambda}\right), \quad F\left(A_{\hat{\beta}'} + \beta_{n+1}, 1 - A_{\hat{\beta}'}, 2A_{\hat{\beta}'} + \beta_{n+1}; \frac{1}{\lambda}\right)$$

are contiguous and irreducible (see Definitions A.1, A.2 respectively). Therefore the hypergeometric function in equation (4.9) is also algebraic over $\mathbb{Q}(\lambda)$ (see Proposition A.2). Now using Lemma 4.1 we conclude that

$$\frac{1}{(2\pi i)^{\frac{n}{2}}}\int_{\delta^{n0}} res\left(\frac{\omega_\beta}{f_n}\right) \in \overline{\mathbb{Q}(\lambda)}.$$

The same reasoning is valid for the integral $\frac{1}{(2\pi i)^{\frac{n}{2}}}\int_{\delta^{n1}} res\left(\frac{\omega_\beta}{f_n}\right)$, which allows us to conclude the result.

■

Remark 4.4. Perhaps it is possible to prove an analogous result for the family $f_n = g_n(x) + P(y)$, where $g_n(x) = x_1^{m_!} + \cdots + x_{n-1}^{m_{n-1}} + x_n^{m_n}$ and $P(y) = y(1-y)(\lambda - y)$ using the similar ideas developed in the appendix of [Del79] by Koblitz and Ogus.

As a consequence of Proposition 4.6, we have that Proposition 4.4 is true in the n-dimensional case not only in the 2-dimensional case. This fact is recorded in the following corollary.

Corollary 4.7. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n^d$ and $P(y) = y(1-y)(\lambda - y)$. Consider $\frac{\omega_\beta}{f}$ a good form (see Definition 3.2) with $A_{\beta'} \notin \mathbb{N}$ and let

$$\delta^0 = \sum_{j=0}^{d-2} n_{0j}\delta_0 * \delta_j^{-1}, \quad \delta^1 = \sum_{j=0}^{d-2} n_{1j}\delta_1 * \delta_j^{-1}.$$

If δ^0 and δ^1 are generic Hodge cycle then either

$$\sum_{j=0}^{d-2} n_{kj}\zeta_d^{j(\beta_2+1)}, \quad k = 0, 1$$

is zero or

$$F\left(A_{\beta'} + \beta_{n+1}, 1 - A_{\beta'}, 2A_{\beta'} + \beta_{n+1}; \frac{1}{\lambda}\right) \text{ and } F\left(A_{\beta'}, 1 - A_{\beta'} - \beta_{n+1}, 2A_{\beta'}; 1 - \lambda\right)$$

are in $\overline{\mathbb{Q}(\lambda)}$.

The argument in Proposition 4.6 does not work for a differential form with pole of order greater than one. However, we can still get a similar result using the same ideas with an extra hypothesis.

Proposition 4.7. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f = g(x) + P(y)$, where $g(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n^d$ and $P(y) = y(1-y)(\lambda - y)$. Consider $\frac{\omega_\beta}{f^k}$ a good form such that $\frac{\omega_\beta}{f}$ is also a good form, and $A_{\beta'} \notin \mathbb{N}$. If $\delta \in H_n(X, \mathbb{Q})$ is a strong generic Hodge cycle, we have

$$\frac{1}{(2\pi i)^{n/2}} \int_\delta res\left(\frac{\omega_\beta}{f^k}\right) \in \overline{\mathbb{Q}(\lambda)}.$$

Proof. Consider $(n_{kj}) \in \mathcal{A}$, this element induces the cycle

$$\delta = \delta^0 + \delta^1 = \sum_{j=0}^{d-2} n_{j,0}\delta_0 * \delta_j^{-1} + \sum_{j=0}^{d-2} n_{j,1}\delta_1 * \delta_j^{-1},$$

with $\delta_j^{-1} \in H_{n-1}(\{g = -1\})$, which in turn induces a strong generic Hodge cycle. We know that δ is a strong generic Hodge cycle if and only if δ^0, δ^1 are strong generic Hodge cycles (see Remark 4.2). By Proposition 3.4, up to multiplication by a nonzero element of $\overline{\mathbb{Q}(\lambda)}$ we have

$$\frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{\delta^0} res\left(\frac{\omega_\beta}{f}\right) = \left(\sum_{j=0}^{d-2} n_{j,0}\zeta_d^{j(\beta_n+1)}\right) \frac{B\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{\beta_n+1}{d}\right) B\left(A_{\beta'}, A_{\beta'}\right)}{(2\pi i)^{\frac{n}{2}}} F\left(a, b, c; \frac{1}{\lambda}\right), \quad (4.11)$$

where $a = A_{\beta'} + \beta_{n+1}$, $b = 1 - A_{\beta'}$, $c = 2A_{\beta'} + \beta_{n+1}$. The fact that $\frac{\omega_\beta}{f}$ is a good form implies that $F\left(a, b, c; \frac{1}{\lambda}\right)$ is irreducible. An inductive argument allows us to prove, up to multiplication by a nonzero element of $\overline{\mathbb{Q}(\lambda)}$ that

$$\begin{aligned}\frac{1}{(2\pi i)^{\frac{n}{2}}} \int_{\delta^0} res\left(\frac{\omega_\beta}{f^k}\right) = &\left(\sum_{j=0}^{d-2} n_{j,0}\zeta_d^{j(\beta_n+1)}\right) \frac{B\left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{\beta_n+1}{d}\right) B\left(A_{\beta'}, A_{\beta'}\right)}{(2\pi i)^{n/2}} \times \\ &\sum_j C_j(\lambda) F\left(a_j, b_j, c_j; \frac{1}{\lambda}\right),\end{aligned} \quad (4.12)$$

with $F\left(a_j, b_j, c_j; \frac{1}{\lambda}\right)$ contiguous to $F\left(a, b, c; \frac{1}{\lambda}\right)$ and $C_j(\lambda) \in \mathbb{Q}(\lambda)$. The first inductive step is for $k = 2$. In this case we use Proposition 3.5. For the general case we apply pole order reduction (see (3.3)) and then the inductive hypothesis. Note that if the integral in (4.11) is zero, then integral in (4.12) is zero. Now, suppose that δ^0 is a strong generic Hodge cycle and that the integral in (4.11) is nonzero. By Proposition 4.6 and Lemma 4.1 we have that $F\left(a, b, c; \frac{1}{\lambda}\right) \in \overline{\mathbb{Q}(\lambda)}$. Therefore $F\left(a_j, b_j, c_j; \frac{1}{\lambda}\right)$ are algebraic over $\mathbb{Q}(\lambda)$ (see Proposition A.2). With this and using Lemma 4.1 we conclude that (4.12) is algebraic over $\mathbb{Q}(\lambda)$. The same is valid for the cycle δ^1. ∎

Remark 4.5. Under the hypotheses of the previous proposition, the proof tells us that the hypergeometric functions that appear in the integral $\frac{1}{(2\pi i)^{n/2}} \int_\delta res\left(\frac{\omega_\beta}{f_n^k}\right)$ are algebraic.

In the 2-dimensional case, the previous result is independent of the hypothesis that $\frac{\omega_\beta}{f}$ is a good form. What will be the nature of the hypergeometric functions that appear in the integral $\frac{1}{2\pi i}\int_{\delta^j} res\left(\frac{\omega_\beta}{f^k}\right)$, $j=0,1$, when $\frac{\omega_\beta}{f^k}$ is a good form and $\frac{\omega_\beta}{f}$ is not a good form? Exploring these integrals with $k=2$ we obtain:

Proposition 4.8. The following expressions are in $\overline{\mathbb{Q}(\lambda)}$:

$$0 \neq 6F\left(\frac{4}{3},-\frac{4}{3},\frac{8}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{2}{3}F\left(\frac{4}{3},-\frac{1}{3},\frac{8}{3};1-\lambda\right)(\lambda+1)(5\lambda^2-8\lambda+5), \tag{4.13}$$

$$0 \neq 2F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)-\frac{2}{3}F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};1-\lambda\right)(\lambda+1), \tag{4.14}$$

$$\begin{aligned}0 \neq &4F\left(\frac{2}{3},-\frac{5}{3},\frac{4}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{1}{3}F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)(\lambda+1)(8\lambda^2-11\lambda+8)+\\ &F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};1-\lambda\right)\lambda(1-\lambda)^2,\end{aligned} \tag{4.15}$$

$$\begin{aligned}0 \neq &6F\left(\frac{2}{3},-\frac{8}{3},\frac{4}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{2}{3}F\left(\frac{2}{3},-\frac{5}{3},\frac{4}{3};1-\lambda\right)(\lambda+1)(7\lambda^2-10\lambda+7)+\\ &2F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)\lambda(1-\lambda)^2,\end{aligned}$$

but each hypergeometric function in the expressions above are not algebraic over $\mathbb{Q}(\lambda)$.

Proof. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f=g(x)+P(y)$, where $g(x)=x_1^2+x_2^6$ and $P(y)=y(1-y)(\lambda-y)$. Consider the good form $\frac{\omega_\beta}{f^2}$ with $\beta=(0,4,0)$. Observe that the form $\frac{\omega_\beta}{f}$ is not a good form. In this case $\mathcal{A}=\mathbb{Q}^{2\times 5}$, so every cycle in $H_2(U,\mathbb{Q})$ induces an element in $\mathrm{SHod}_2(X,\mathbb{Q})_0$. Now consider the strong generic Hodge cycle induced by $\delta^1=\sum_{j=0}^4 n_j\delta_1 * \delta_j^{-1}$ with $n_0-n_3+n_2\neq 0$ or $n_1-n_2+n_4\neq 0$. This guarantees that

$$\sum_{j=0}^{4} n_j\zeta_6^{5j}\neq 0,$$

since variety X is 2-dimensional, we have

$$\frac{1}{2\pi i}\int_{\delta^1} res\left(\frac{\omega_\beta}{f^2}\right)\in\overline{\mathbb{Q}(\lambda)}.$$

Therefore by Proposition 3.5 and Lemma 4.1 we conclude that

$$\left[6F\left(\frac{4}{3},-\frac{4}{3},\frac{8}{3};1-\lambda\right)(\lambda^2-\lambda+1)-\frac{2}{3}F\left(\frac{4}{3},-\frac{1}{3},\frac{8}{3};1-\lambda\right)(\lambda+1)(5\lambda^2-8\lambda+5)\right]\in\overline{\mathbb{Q}(\lambda)}.$$

It remains to prove that $F\left(\frac{4}{3},-\frac{4}{3},\frac{8}{3};1-\lambda\right)$, $F\left(\frac{4}{3},-\frac{1}{3},\frac{8}{3};1-\lambda\right)\notin\overline{\mathbb{Q}(\lambda)}$. For this, note that the above hypergeometric functions are reducible, so use Theorem A.4. To obtain the other expressions the reasoning is the same but using the differential forms $\frac{\omega_\beta}{f^2}$ with $\beta=(0,0,\beta_3)$, and $\beta_3=0,1,2$.

■

Remark 4.6. Let X be a desingularization of the weighted hypersurface D given by the quasi-homogenization F of $f=g(x)+P(y)$, where $g(x)=x_1^2+x_2^d$, $6|d$ and $P(y)=y(1-y)(\lambda-y)$. If for each $\beta_2\in\left\{\frac{5d}{6}-1,\frac{d}{6}-1\right\}$, there is a strong generic Hodge cycle $\delta^1=\sum_{j=0}^{d-2}n_j\delta_1*\delta_j^{-1}$ on X such that

$$\sum_{j=0}^{d-2}n_j\zeta_d^{j(\beta_2+1)}\neq 0,$$

then we obtain exactly the same result of Proposition 4.8 using the differential forms $\frac{\omega_\beta}{f^2}$ with $\beta=\left(0,\frac{5d}{6}-1,0\right)$ and $\beta=\left(0,\frac{d}{6}-1,\beta_3\right)$, $\beta_3=0,1,2$.

Using the same idea with the same β's in the same variety of the proof of Proposition 4.8 and with the cycle $\delta^0=\sum_{j=0}^{4}n_j\delta_0*\delta_j^{-1}$ such that $n_0-n_3+n_2\neq 0$ or $n_1-n_2+n_4\neq 0$, we have

Proposition 4.9. The following expressions are in $\overline{\mathbb{Q}(\lambda)}$:

$$0\neq\frac{3}{5}F\left(\frac{7}{3},-\frac{1}{3},\frac{11}{3};\frac{1}{\lambda}\right)(\lambda^2-\lambda+1)-\frac{2}{15}F\left(\frac{4}{3},-\frac{1}{3},\frac{8}{3};\frac{1}{\lambda}\right)(\lambda+1)\left(5\lambda^2-8\lambda+5\right),$$

$$0\neq F\left(\frac{5}{3},\frac{1}{3},\frac{7}{3};\frac{1}{\lambda}\right)-\frac{2}{3}F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};\frac{1}{\lambda}\right)(\lambda+1),$$

$$0\neq\frac{10}{7}F\left(\frac{8}{3},\frac{1}{3},\frac{10}{3};\frac{1}{\lambda}\right)(\lambda^2-\lambda+1)-\frac{1}{6}F\left(\frac{5}{3},\frac{1}{3},\frac{7}{3};\frac{1}{\lambda}\right)(\lambda+1)\left(8\lambda^2-11\lambda+8\right)+ F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};\frac{1}{\lambda}\right)\lambda(1-\lambda)^2,$$

$$0\neq\frac{24}{7}F\left(\frac{11}{3},\frac{1}{3},\frac{13}{3};\frac{1}{\lambda}\right)(\lambda^2-\lambda+1)-\frac{10}{21}F\left(\frac{8}{3},\frac{1}{3},\frac{10}{3};\frac{1}{\lambda}\right)(\lambda+1)\left(7\lambda^2-10\lambda+7\right)+ 2F\left(\frac{5}{3},\frac{1}{3},\frac{7}{3};\frac{1}{\lambda}\right)\lambda(1-\lambda)^2,$$

but each hypergeometric function in the expressions above is not algebraic over $\mathbb{Q}(\lambda)$.

Remark 4.7. The algebraic functions of the expressions in Propositions 4.8 and 4.9 can be found using Gauss' relations. For example using the relation

$$(c-b)F(a,b-1,c;z)+(2b-c-bz+az)F(a,b,c;z)+b(z-1)F(a,b+1,c;z)=0, \tag{4.16}$$

with $a=\frac{2}{3}$, $b=\frac{1}{3}$ and $c=\frac{4}{3}$ we obtain that equation (4.14) is equal to $\frac{2}{3}\lambda^{\frac{1}{3}}$. Using the latter together with equation (4.16) where $a=\frac{2}{3}$, $b=\frac{-2}{3}$ and $c=\frac{4}{3}$ we find that equation (4.15) is equal to $\frac{1}{3}\lambda^{\frac{4}{3}}(\lambda+1)$.

Remark 4.8. The differential forms used in Propositions 4.8 and 4.9 are all forms such that $\frac{\omega_\beta}{f^2}$ is a good form, $\frac{\omega_\beta}{f}$ is not a good form, with $A_\beta < 2$ and $A_{\beta'} \notin \mathbb{N}$. We would like to get more algebraic expressions of hypergeometric functions such that the hypergeometric functions are not algebraic. One possible path would be to explore the integrals of good forms $\frac{\omega_\beta}{f^2}$ with $\frac{\omega_\beta}{f}$ is not good form, $A_\beta > 2$ and $A_{\beta'} \notin \mathbb{N}$. The following proposition tells us that such a path is not possible.

Proposition 4.10. Consider $f = g(x)+P(y)$, where $g(x) = x_1^{m_1}+\cdots+x_n^{m_n}$ and $P(y) = y(1-y)(\lambda-y)$. Suppose that $\frac{\omega_\beta}{f^k}$ is a good form with $A_\beta > k$, then $\frac{\omega_\beta}{f^{k-1}}$ is a good form.

Proof. Since $\frac{\omega_\beta}{f^k}$ is a good form, we can write

$$\frac{\omega_\beta}{f^k} = \sum C_{k_j} \frac{\omega_{\beta+(0',k_j)}}{f^j}, \tag{4.17}$$

with $C_{k_j} \in \mathbb{C}[\lambda]$ and $A_{\beta+(0',k_j)} = A_\beta + \frac{k_j}{3} < j$ such that $j-1 < A_\beta + \frac{k_j-1}{3}$ or $j-1 < A_\beta + \frac{k_j-2}{3}$ (see Remark 3.5). Now let us apply the process of pole order increment to the differential form $\frac{\omega_\beta}{f^{k-1}}$. We obtain

$$\frac{\omega_\beta}{f^{k-1}} = \sum \hat{C}_{k_j} \frac{\omega_{\beta+(0',k_j)}}{f^{j-1}},$$

where $j-1 < A_{\beta+(0',k_j)} < j$. This means that we need to increment the pole order again. Let us see what happens when $j-1 < A_\beta + \frac{k_j-1}{3}$. We have

$$\left[\frac{\omega_{\beta+(0',k_j)}}{f^{j-1}}\right] = \frac{A_\beta + \frac{k_j}{3}}{3(A_\beta + \frac{k_j}{3} - j)} \left[\frac{-a\omega_{\beta+(0',k_j+2)} - 2b\omega_{\beta+(0',k_j+1)}}{f^j}\right].$$

We must analyze each term of the previous expression. Let us see the most problematic term: $\frac{\omega_{\beta+(0',k_j+1)}}{f^j}$. Observe that $j - \frac{1}{3} < A_{\beta+(0',k_j+1)} < j + \frac{1}{3}$. If $A_{\beta+(0',k_j+1)} = j$, then $A_{\beta+(0',k_j-2)} = j-1$. This implies that $\frac{\omega_\beta}{f^k}$ is not good form or the differential form $\frac{\omega_{\beta+(0',k_j-1)}}{f^{j-1}}$ appears one step before obtaining equation (4.17). If we have $\frac{\omega_{\beta+(0',k_j-1)}}{f^{j-1}}$ as $A_{\beta+(0',k_j-1)} = j - \frac{2}{3}$ we need to apply the process of pole order increment again but by applying it we get that $\frac{\omega_\beta}{f^k}$ is not a good form because appears the differential form $\frac{\omega_{\beta+(0',k+1)}}{f^j}$ and $A_{\beta+(0',k_j+1)} = j$. In conclusion $A_{\beta+(0',k_j+1)} \neq j$. If necessary we increment the pole order again. The other cases are similar, leading us to conclude that $\frac{\omega_\beta}{f^{k-1}}$ is a good form. ■

Remark 4.9. Proposition 4.10 tells us that in Proposition 4.7, the condition that $\frac{\omega_\beta}{f}$ is a good form is not necessary for $k \leq \frac{n}{2} - 1$.

To obtain more algebraic expressions of hypergeometric functions such that the hypergeometric functions are not algebraic we integrate a strong generic Hodge cycle in a good form $\frac{\omega_\beta}{f^k}$ such that $\frac{\omega_\beta}{f}$ is not good form, $A_\beta < k$ and $A_{\beta'} \notin \mathbb{N}$, where $f = g(x)+P(y)$, $g(x) = x_1^2+x_2^d$ and $P(y) = y(1-y)(\lambda-y)$. Indeed in the proof of Proposition 4.7 we saw that up to algebraic element over $\mathbb{Q}(\lambda)$

$$\frac{1}{2\pi i}\int_{\delta^0} res\left(\frac{\omega_\beta}{f^k}\right) = \frac{B\left(\frac{1}{2},\frac{\beta_2+1}{d}\right)B\left(A_{\beta'},A_{\beta'}\right)}{2\pi i}\sum_j C_j(\lambda)F\left(a_j,b_j,c_j;\frac{1}{\lambda}\right), \tag{4.18}$$

with $F\left(a_j,b_j,c_j;\frac{1}{\lambda}\right)$ contiguous to $F\left(a,b,c;\frac{1}{\lambda}\right)$, where $a = A_{\beta'}+\beta_3$, $b = 1-A_{\beta'}$, $c = 2A_{\beta'}+\beta_3$. Therefore if $\delta^0 = \sum_{j=0}^{d-2} n_j\delta_0 * \delta_j^{-1}$ is a generic Hodge cycle we have that (4.18) belongs to $\overline{\mathbb{Q}(\lambda)}$. Furthermore, if

$$\sum_{j=0}^{d-2} n_j\zeta_d^{j(\beta_2+1)} \neq 0,$$

using Lemma 4.1 we conclude that $\sum_j C_j(\lambda)F\left(a_j,b_j,c_j;\frac{1}{\lambda}\right) \in \overline{\mathbb{Q}(\lambda)}$. The fact that $\frac{\omega_\beta}{f}$ is not a good form implies that $A_{\beta'} = \frac{N}{3}$ for some $N \in \mathbb{N}$, and therefore $F\left(a,b,c;\frac{1}{\lambda}\right)$ is reducible. Also note that

$$a_j = a + k_j, \quad b_j = b + l_j, \quad c_j = c + d_j$$

with $k_j, l_j, d_j \in \mathbb{Z}$. Consider $\lambda_j = 1-c_j$, $\mu_j = c_j - a_j - b_j$, $\nu_j = a_j - b_j$. A straightforward computation allows us to verify that λ_j, μ_j, ν_j do not satisfy the hypothesis of Theorem A.4 and therefore $F\left(a_j,b_j,c_j;\frac{1}{\lambda}\right) \notin \overline{\mathbb{Q}(\lambda)}$. The same is valid for the cycle δ^1.

4.5 Computational verification

We can check the validity of Propositions 4.8 and 4.9 using numerical computations by evaluating λ at algebraic numbers. Call $G(\lambda)$ the function defined by equation (4.13). We use the package `with(IntegerRelations)` in **Maple**. The command

```
v := expand([seq(evalf[k](G(lambda)^j), j = 0 .. m)]);
```

computes powers of $G(\lambda)$ from 0 to m with k digits of precision. With the following command

```
u := LinearDependency(v, method = LLL);
```

we find a $\mathbb{Z}$-linear relation between $1, G(\lambda), G(\lambda)^2, \ldots, G(\lambda)^m$. The polynomial that satisfies $G(\lambda)$ can be displayed with the command

```
P := add(u[j]*z^(j-1), j = 1 .. m+1);
```

This computation is heuristic, since we only have approximations of $G(\lambda)$. As an example of the above take $\lambda = i$ with $i^2 = -1$, $m = 400$ and 400 digits of precision. We have the polynomial

$$81z^4 - 900z^2 + 10000.$$

These computations suggest that $G(i)$ is an algebraic number. This is, of course, one consequence of Proposition 4.8. On the other hand, using the same ideas from Remark 4.7, we can see that $G(\lambda) = \frac{10}{3}\lambda^{\frac{5}{3}}$. Observe that $G(i)$ is root of $81z^4 - 900z^2 + 10000$.

With these same commands we can verify what was proven by Reiter and Movasati in [MR06] mentioned in the introduction of this thesis. Let us remember it: they proved that

$$e^{-\frac{5}{6}\pi i}\frac{F\left(\frac{5}{6},\frac{1}{6},1;\frac{27}{16}t^2\right)}{F\left(\frac{5}{6},\frac{1}{6},1;1-\frac{27}{16}t^2\right)} \tag{4.19}$$

belongs to $\mathbb{Q}(\zeta_3)$ for some $t\in\overline{\mathbb{Q}}$ if and only if

$$G(t)=\pi^2\frac{F\left(\frac{5}{6},\frac{1}{6},1;\frac{27}{16}t^2\right)}{\Gamma\left(\frac{1}{3}\right)^3},\quad F(t)=\pi^2\frac{F\left(\frac{5}{6},\frac{1}{6},1;1-\frac{27}{16}t^2\right)}{\Gamma\left(\frac{1}{3}\right)^3}\in\overline{\mathbb{Q}}.$$

We have that t satisfies

$$91125t^4-54000t^2+256=0 \quad\text{or}\quad 81000t^4-48000t^2-1=0, \tag{4.20}$$

then equation (4.19) belongs to $\mathbb{Q}(\zeta_3)$ (see [MR06] and the references therein). Therefore the result of Movasati and Reiter predicts that for t satisfying (4.20), $G(t)$ and $F(t)$ are algebraic. We can check the validity of the above by numerical calculation in the same way that we did at the beginning of this section. Let us remember the commands: we use the package `with(IntegerRelations)` in **Maple**. The command

```
alias(a=RootOf(91125*t^4-54000*t^2+256));
```

take a root a of the polynomial $91125t^4-54000t^2+256$. The following command

```
v:=expand([seq(evalf[k](G(a)^j), j = 0 .. m)]);
```

compute powers of $G(a)$ from 0 to m with k digits of precision. Finally with the following command

```
u:=LinearDependency(v,method=LLL);
```

we find a $\mathbb{Z}$-linear relation between $1,G(a),G(a)^2,\dots,G(a)^m$. The polynomial that satisfies $G(a)$ can be displayed with the command

```
P:=add(u[j]*z^(j-1), j = 1 .. m+1);
```

Therefore with $m = 600$ and 600 digits of precision we have the polynomial

$$
\begin{aligned}
&-24z^{103} - 66z^{102} - 469z^{101} + 4598z^{100} - 24076z^{99} + 308554z^{98} + 257239z^{97} + 86683z^{96} + 944482z^{95} \\
&- 1692348z^{94} + 2393334z^{93} + 2157806z^{92} - 833641z^{91} - 2433103z^{90} - 2539863z^{89} + 9457654z^{88} + \\
&5670219z^{87} - 10724295z^{86} + 12611619z^{85} + 20099959z^{84} + 32736062z^{83} - 4201482z^{82} - 12032058z^{81} \\
&- 5889683z^{80} + 6318082z^{79} - 4948710z^{78} + 10536701z^{77} - 22909707z^{76} + 19232408z^{75} \\
&- 44161749z^{74} + 4357271z^{73} + 17670607z^{72} - 24033795z^{71} + 19070447z^{70} - 12030126z^{69} \\
&- 29262478z^{68} + 58699595z^{67} + 13483109z^{66} - 2773158z^{65} + 26752002z^{64} - 419533z^{63} + \\
&29904688z^{62} + 10394590z^{61} - 11699693z^{60} - 41479848z^{59} + 15195839z^{58} + 8851662z^{57} - 34545314z^{56} \\
&- 43883797z^{55} + 27026236z^{54} + 26900260z^{53} - 6421512z^{52} + 17391813z^{51} - 6799741z^{50} \\
&- 5453700z^{49} - 29636969z^{48} - 4265921z^{47} - 18119977z^{46} + 69776238z^{45} - 17004057z^{44} + \\
&37416041z^{43} - 30938342z^{42} + 42190945z^{41} + 37539982z^{40} - 7235067z^{39} - 3719318z^{38} - 12425103z^{37} \\
&- 48524284z^{36} - 6509682z^{35} - 26346217z^{34} + 17800611z^{33} - 66311884z^{32} - 46619350z^{31} + \\
&21324186z^{30} - 116081494z^{29} - 39187149z^{28} + 88746134z^{27} + 12865846z^{26} + 86679407z^{25} + \\
&220123189z^{24} - 99134066z^{23} - 101117505z^{22} - 19528725z^{21} - 66938006z^{20} + 56153740z^{19} \\
&- 1396313z^{18} + 69914142z^{17} + 118649131z^{16} - 38751206z^{15} + 31885777z^{14} - 87759961z^{13} + \\
&26172790z^{12} - 12363013z^{11} - 20314144z^{10} + 18062061z^{9} - 37016836z^{8} - 20181015z^{7} + 15511996z^{6} \\
&- 13485267z^{5} + 26817965z^{4} - 30238241z^{3} - 81138068z^{2} + 10458592z + 18848508.
\end{aligned}
$$

These computations suggests that $G(a)$ with a root of $91125t^4 - 54000t^2 + 256$ is an algebraic number. This is, of course, a consequence of the result of Movasati and Reiter stated above.

APPENDIX A

Hypergeometric function

In our context, naturally the hypergeometric function appears when calculating periods (hence the name of hypergeometric periods). That's why this chapter contains a summary of the properties of the hypergeometric function. For a more complete exposition see [Yos13, Beu07, AS48].

A.1 Hypergeometric series

Let us define the hypergeometric series by the power series

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := a(a+1)\cdots(a+n-1),$$

and c is not $0, -1, -2, \ldots$. This function is symmetric in a and b, and the radius of convergence of this series es 1, unless a or b is a nonpositive integer, in which case the function is a polynomial

$$F(-m,b,c;z) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \frac{(b)_n}{(c)_n} z^n, \quad m > 0.$$

The holomorphic function defined by the hypergeometric series, as well as its analytic continuation, is called the **(Gauss) hypergeometric function**. The hypergeometric function satisfies a linear differential equation, namely

$$E(a,b,c) : z(1-z)\frac{d^2u}{dz^2} + \{c - (a+b+1)z\}\frac{du}{dz} - abu = 0. \tag{A.1}$$

This equation is called the **hypergeometric (differential) equation.**

Proposition A.1. If $Re(c) > Re(a) > 0$ then we have

$$F(a,b,c;z) = \frac{1}{B(a,c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt, \tag{A.2}$$

where B is the beta function. The integral represents a one valued analytic function in the z-plane cut along the real axis from 1 to ∞. Therefore (A.2) gives the analytic continuation of $F(a,b,c;z)$. Another integral representation is in the form of a Mellin-Barnes integral

$$F(a,b,c;z) = \frac{\Gamma(c)}{2\pi i \Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds; \quad |arg(-z)| < \pi.$$

The path of integration is curved, if necessary, to separate the poles $s = -a-n$ and $s = -b-n$ from the poles $s = n$ with $n \in \mathbb{N}$. The cases in which $-a$, $-b$ or $-c$ are non-negative integers or $a-b$ equal to an integer are excluded.

A.2 Kummer's 24 solutions

From the last two equations a number of transformations for hypergeometric function can be derived

$$F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z) \tag{A.3}$$

$$F(a,b,c;z) = (1-z)^{-a}F\left(a,c-b,c;\frac{z}{z-1}\right). \tag{A.4}$$

If none of the numbers $c,\ c-a-b,\ a-b$ is equal to an integer, then two linearly independent solutions of hypergeometric equation (A.1) in the neighborhood of the singular points $0,\ 1,\ \infty$ are respectively

$$u_{1(0)} = F(a,b,c;z) = (1-z)^{c-a-b}F(c-a,c-b,c;z) \tag{A.5}$$

$$u_{2(0)} = z^{1-c}F(a-c+1,b-c+1,2-c;z) = z^{1-c}(1-z)^{c-a-b}F(1-a,1-b,2-c;z) \tag{A.6}$$

$$u_{1(1)} = F(a,b,a+b+1-c;1-z) = z^{1-c}F(1+b-c,1+a-c,a+b+1-c;1-z) \tag{A.7}$$

$$u_{2(1)} = (1-z)^{c-a-b}F(c-b,c-a,c-a-b+1;1-z) = z^{1-c}(1-z)^{c-a-b}F(1-a,1-b,c-a-b+1;1-z) \tag{A.8}$$

$$u_{1(\infty)} = z^{-a}F(a,a-c+1,a-b+1;z^{-1}) = z^{b-c}(z-1)^{c-a-b}F(1-b,c-b,a-b+1;z^{-1}) \tag{A.9}$$

$$u_{2(\infty)} = z^{-b}F(b,b-c+1,b-a+1;z^{-1}) = z^{a-c}(z-1)^{c-a-b}F(1-a,c-a,b-a+1;z^{-1}). \tag{A.10}$$

Where the second set of equalities are obtained by applying the equation (A.3) to the first set of expressions. Now applying equation (A.4) from equation (A.5) to (A.10) we obtain another set of representations for the previous solutions

$$u_{1(0)} = (1-z)^{-a}F\left(a,c-b,c;\frac{z}{z-1}\right) = (1-z)^{-b}F\left(b,c-a,c;\frac{z}{z-1}\right) \tag{A.11}$$

$$u_{2(0)} = z^{1-c}(1-z)^{c-a-1}F\left(a-c+1, 1-b, 2-c; \frac{z}{z-1}\right)$$
$$= z^{1-c}(1-z)^{c-b-1}F\left(b-c+1, 1-a, 2-c; \frac{z}{z-1}\right) \quad \text{(A.12)}$$

$$u_{1(1)} = z^{-a}F(a, a-c+1, a+b+1-c; 1-z^{-1}) = z^{-b}F(b, b+1-c, a+b+1-c; 1-z^{-1}) \quad \text{(A.13)}$$

$$u_{2(1)} = z^{a-c}(1-z)^{c-a-b}F(c-a, 1-a, c-a-b+1; 1-z^{-1}) = z^{b-c}(1-z)^{c-a-b}F(c-b, 1-b, c-a-b+1; 1-z^{-1}) \quad \text{(A.14)}$$

$$u_{1(\infty)} = (z-1)^{-a}F\left(a, c-b, a-b+1; (1-z)^{-1}\right) = (z-1)^{-b}F(b, c-a, b-a+1; (1-z)^{-1}) \quad \text{(A.15)}$$

$$u_{2(\infty)} = z^{1-c}(z-1)^{c-a-1}F\left(a-c+1, 1-b, a-b+1; \frac{1}{1-z}\right)$$
$$= z^{1-c}(z-1)^{c-b-1}F\left(b-c+1, 1-a, b-a+1; \frac{1}{1-z}\right). \quad \text{(A.16)}$$

The set of equations from (A.5) to (A.16) are called **Kummer's 24 solutions** of the hypergeometric equation.

Definition A.1. We call any function $F(a+k, b+l, c+m; z)$ with $k, l, m \in \mathbb{Z}$ **contiguous** with $F(a, b, c; z)$.

Gauss found that among three contiguous hypergeometric functions F_1, F_2 and F_3 there exists a relation of the form $a_1F_1 + a_2F_2 + a_3F_3 = 0$, where a_1, a_2 and a_3 are rational functions of z. These relations are known as **Gauss' relations**, for example:

$$c(c-1)(z-1)F(a, b, c-1; z) + c[c-1-(2c-a-b-1)z]F(a, b, c; z) + (c-a)(c-b)zF(a, b, c+1; z) = 0$$

Other relations that the hypergeometric function satisfies are:

$$\frac{d^n}{dz^n}F(a, b, c; z) = \frac{(a)_n(b)_n}{(c)_n}F(a+n, b+n, c+n; z) \quad \text{(A.17)}$$

$$\frac{d^n}{dz^n}\left[z^{a+n-1}F(a, b, c; z)\right] = (a)_n z^{a-1}F(a+n, b, c; z) \quad \text{(A.18)}$$

$$\frac{d^n}{dz^n}\left[z^{c-a+n-1}(1-z)^{a+b-c}F(a, b, c; z)\right] = (c-a)_n z^{c-a-1}(1-z)^{a+b-c-n}F(a-n, b, c; z) \quad \text{(A.19)}$$

$$\frac{d^n}{dz^n}\left[(1-z)^{a+b-c}F(a, b, c; z)\right] = \frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}F(a, b, c+n; z) \quad \text{(A.20)}$$

$$\frac{d^n}{dz^n}\left[z^{c-1}F(a, b, c; z)\right] = (c-n)_n z^{c-n-1}F(a, b, c-n; z). \quad \text{(A.21)}$$

So, the above equations allow us to deduce the following:

Proposition A.2. Suppose $a, b \not\equiv 0, c \ (mod\ \mathbb{Z})$ and $c \notin \mathbb{Z}$. Then each function $F(a+k, b+l, c+m; z)$ is equal a linear combination of F and F' with rational functions as coefficients, where $k, l, m \in \mathbb{Z}$. In particular any three contiguous functions satisfy a $\mathbb{C}(z)$-linear relation. In particular each $F(a, b, c; z)$ contiguous to an algebraic Hypergeometric function is algebraic.

A.3 Monodromy

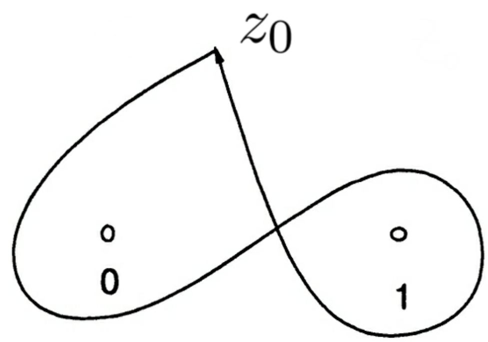

Figure A.1: A loop γ

The hypergeometric differential equation

$$E(a,b,c) : z(1-z)\frac{d^2u}{dz^2} + \{c-(a+b+1)z\}\frac{du}{dz} - abu = 0.$$

is linear, of second order and has singularities at $z = 0, 1, \infty$. For any $z_0 \in \mathbb{C}\backslash\{0,1\} = \mathbb{P}^1\backslash\{0,1,\infty\}$ there are two linearly independent analytic solutions f_1, f_2 around z_0. These solutions can be analytically continued along any path in $\mathbb{C}\backslash\{0,1\}$. If γ is a loop in $\mathbb{C}\backslash\{0,1\}$ starting and ending at z_0, the analytic continuation of f_1, f_2 along γ, $\gamma_* f_1, \gamma_* f_2$, are again solutions of the equation $E(a,b,c)$ around z_0. Hence there exists $M(\gamma) \in GL(2,\mathbb{C})$ such that

$$\gamma_* \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = M(\gamma) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

This induces the group homomorphism

$$\begin{array}{ccc} \pi_1(\mathbb{C}\backslash\{0,1\}, z_0) & \longrightarrow & GL(2,\mathbb{C}) \\ \gamma & \longmapsto & M(\gamma), \end{array}$$

which is called **the monodromy representation** of the differential equation. **The monodromy group** is the image of the monodromy representation. The above map depends on the choice of f_1, f_2, z_0. If we take another two linearly independent solutions, the new representation is conjugated to the old one. This is also the case when we change z_0, the new monodromy group is conjugated to the old one. So the differential equation determines the conjugacy class of a monodromy representation and its monodromy group.

Definition A.2. The hypergeometric equation $E(a,b,c)$ is called **irreducible** if neither of a, b, $c-a$, $c-b$ is an integer.

Now, we assume that $a, b, c \in \mathbb{R}$. Let z_0 be a point in the upper half plane $\mathcal{H}$ and let f_1, f_2 be two independent solutions of the hypergeometric equation $E(a, b, c)$ around to z_0. The map

$$\begin{array}{ccc} \mathcal{H} & \overset{D(z)}{\longrightarrow} & \mathbb{P}^1 \\ z & \longmapsto & [f_0(z) : f_1(z)] \end{array}$$

is called **Schwarz map** and satisfies:

Theorem A.1 (Schwarz)**.** Let $\lambda = |1-c|$, $\mu = |c-a-b|$, $\nu = |a-b|$ such that $0 \leq \lambda, \mu, \nu < 1$. Then $D(z)$ maps $\mathcal{H} \cup \mathbb{R}$ one-to-one onto curvilinear triangle. The vertices correspond to the points $D(0)$, $D(1)$, $D(\infty)$ and the corresponding angles are $\lambda\pi$, $\mu\pi$, $\nu\pi$.

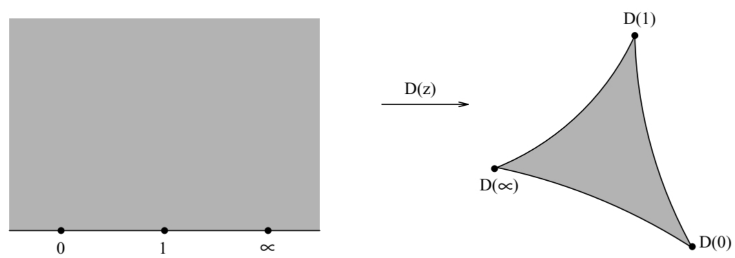

Figure A.2: Schwarz map

The monodromy group of the hypergeometric equation $E(a, b, c)$ can be described as follows: Let W be the group generated by the reflections along the edges of the curvilinear triangle which is the image of $\mathcal{H}$ by Schwarz map $D(z)$. The monodromy group of the hypergeometric equation is isomorphic to the subgroup of W consisting of all elements which are product of an even number of reflections. The following well-known theorem indicates the importance of the monodromy group to determine when the hypergeometric equation $E(a, b, c)$ has algebraic solutions, see [Sch73], [vdW02, Theorems 1.7.1, 2.1.8].

Theorem A.2. The following statements are equivalent

i) The hypergeometric equation $E(a, b, c)$ has a basis (y_1, y_2) of algebraic solutions.

ii) The monodromy group of the hypergeometric equation is finite.

iii) $\frac{y_1}{y_2}$ is algebraic and a, b, c are rational.

Schwarz was the first to study and classify hypergeometric equations with finite monodromies in [Sch73]. He showed that only the exponent differences λ, μ, ν are of importance for a hypergeometric equation to have an algebraic ratio of solutions $\frac{y_1}{y_2}$. Moreover, Schwarz gave all triples (λ, μ, ν) in a list of 15 types of these hypergeometric equations, divided by the isomorphism class of the monodromy group, see table A.1.

Number	(λ, μ, ν)	Monodromy group
1	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n}\right)$	Dihedral
2	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)$	Tetrahedral
3	$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$	Tetrahedral
4	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$	Octahedral
5	$\left(\frac{2}{3}, \frac{1}{4}, \frac{1}{4}\right)$	Octahedral
6	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)$	Icosahedral
7	$\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{5}\right)$	Icosahedral
8	$\left(\frac{1}{2}, \frac{1}{5}, \frac{2}{5}\right)$	Icosahedral
9	$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right)$	Icosahedral
10	$\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{5}\right)$	Icosahedral
11	$\left(\frac{2}{3}, \frac{1}{5}, \frac{1}{5}\right)$	Icosahedral
12	$\left(\frac{1}{3}, \frac{2}{5}, \frac{3}{5}\right)$	Icosahedral
13	$\left(\frac{1}{3}, \frac{1}{5}, \frac{3}{5}\right)$	Icosahedral
14	$\left(\frac{1}{5}, \frac{1}{5}, \frac{4}{5}\right)$	Icosahedral
15	$\left(\frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right)$	Icosahedral

Table A.1: Schwarz's list

Theorem A.3 (Schwarz [Sch73]). Let $F(a,b,c;z)$ be a solution of a irreducible hypergeometric equation $E(a,b,c)$ such that $0 \leq \lambda, \mu, \nu < 1$, $0 \leq \lambda + \mu,\ \lambda + \nu,\ \mu + \nu \leq 1$ with $\lambda = |1-c|$, $\mu = |c-a-b|$, $\nu = |a-b|$. Then $F(a,b,c;z)$ is algebraic over $\mathbb{C}(z)$ if only if (λ, μ, ν) is in the table A.1

For this version see [Bat53, §2.7.2]. For other versions see [Żoł06, §12.17] and [Kim69, §5]. By the previous theorem and by Proposition A.2 we have

Corollary A.1. Suppose $a, b \not\equiv 0, c\ (mod\ \mathbb{Z})$ and $c \notin \mathbb{Z}$. If $F(a,b,c;z)$ is contiguous to a hypergeometric function with angular parameters (λ, μ, ν) belonging to the Schwarz's list (table A.1), then $F(a,b,c;z)$ is algebraic.

There is a not well-known result that characterizes when a reducible hypergeometric equation has a basis of algebraic solutions, see [Żoł06, §12.17] and [Kim69, §5].

Theorem A.4 (Schwarz [Sch73]). Consider $\lambda = 1-c$, $\mu = c-a-b$, $\nu = a-b$. The reducible hypergeometric equation $E(a,b,c)$ has a basis of algebraic solutions if and only if none of the singular points $z = 0, 1, \infty$ is logarithmic and exactly two of the numbers $\lambda+\mu+\nu,\ -\lambda+\mu+\nu,\ \lambda-\mu+\nu,\ \lambda+\mu-\nu$ are odd integers.

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