Hodge cycles and Gauss hypergeometric function

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Seminar GADEPs
Geometry, Arithmetic and Differential equations of Periods.
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Theorem

Let X_{P_0} be desingularization of the weighed hypersurface D given by F homogenization of $f = g(x) + P_0(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$ and $P_0(y)$ is a polynomial of degree m.

1. *For $m_1 = \cdots = m_{n-1} = 2$ and $m \geq 7$, we have*

$$dimSGHC_n(X_{P_0}) \leq \begin{cases} m-1 & m_n \text{ even,} \\ 0 & m_n \text{ odd.} \end{cases}$$

2. *For $m_1 = \cdots = m_{n-2} = 2$, m_{n-1} prime, $(m_{n-1}, m_n) = 1$ and $\frac{1}{m_{n-1}} + \frac{1}{m} < \frac{1}{2}$, we have $SGHC_n(X_{P_0}, \mathbb{Q}) \cong 0$.*
3. *For m_j different prime numbers, we have $SGHC_n(X_{P_0}, \mathbb{Q}) \cong 0$*

Proposition

The following expressions are in $\overline{\mathbb{Q}(\lambda)}$ *:*

$$\begin{aligned} 0 \neq & 6F\left(\frac{4}{3}, -\frac{4}{3}, \frac{8}{3}; 1-\lambda\right)(\lambda^2-\lambda+1) \\ & -\frac{2}{3}F\left(\frac{4}{3}, -\frac{1}{3}, \frac{8}{3}; 1-\lambda\right)(\lambda+1)(5\lambda^2-8\lambda+5), \end{aligned}$$

$$\begin{aligned} 0 \neq & 4F\left(\frac{2}{3}, -\frac{5}{3}, \frac{4}{3}; 1-\lambda\right)(\lambda^2-\lambda+1) \\ & -\frac{1}{3}F\left(\frac{2}{3}, -\frac{2}{3}, \frac{4}{3}; 1-\lambda\right)(\lambda+1)(8\lambda^2-11\lambda+8) \\ & +F\left(\frac{2}{3}, \frac{1}{3}, \frac{4}{3}; 1-\lambda\right)\lambda(1-\lambda)^2, \end{aligned}$$

but each hypergeometric function in the expressions above are not algebraic over $\overline{\mathbb{Q}(\lambda)}$.

Consider $f(x,y) = g(x) + P(y)$ where $g = x_1^{m_1} + \cdots + x_n^{m_n}$ and $P(y)$ polynomial with discriminant nonzero.

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where F is the homogenization of f given by

$$F(x_0, \ldots, x_{n+1}) = x_0^d f\left(\frac{x_1}{x_0^{v_1}}, \ldots, \frac{x_{n+1}}{x_0^{v_{n+1}}}\right),$$

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with $v = (v_1, \ldots, v_{n+1})$ and $v_j = \frac{d}{m_j}$ where d is the least common multiple of $m_1, \ldots, m_{n+1}$.

Consider $f(x,y)=g(x)+P(y)$ where $g=x_1^{m_1}+\cdots+x_n^{m_n}$ and $P(y)$ polynomial with discriminant nonzero.

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$$U \subset D \subset X.$$

Hodge Cycles

Definition

$$Hodge_n(X,\mathbb{Q})_0 := \frac{\left\{\delta \in H_n(U,\mathbb{Q}) \mid \int_\delta res\left(\frac{\omega_\beta}{f^j}\right) = 0, A_\beta < j, 1 \leq j \leq \frac{n}{2}\right\}}{\left\{\delta \in H_n(U,\mathbb{Q}) \mid \int_\delta res\left(\frac{\omega_\beta}{f^j}\right) = 0, A_\beta < j, 1 \leq j \leq n+1\right\}},$$

with $U \subset D \subset X$, $\omega_\beta = x^\beta dx := x_1^{\beta_1} \dots x_{n+1}^{\beta_{n+1}} dx_1 \wedge \cdots \wedge dx_{n+1}$ and $A_\beta = \sum_{j=1}^{n+1} \frac{\beta_j+1}{m_j}$

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- $H^{n+1}_{dR}(\mathbb{C}^{n+1}\backslash U) \overset{res}{\to} H^{n}_{dR}(U)$ is an isomorphism.

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$$\frac{\omega_\beta}{f^j} \rightsquigarrow \frac{\omega}{f}$$

$$\left[\frac{\omega_\beta}{f^j}\right] = \frac{1}{\Delta}\left[\frac{\beta_{n+1}Q_1}{j-1}\frac{\omega_{\beta-(0,1)}}{f^{j-1}} + \left(\left(1-\frac{A_{\beta'}}{j-1}\right)Q_2 + \frac{Q_1'}{j-1}\right)\frac{\omega_\beta}{f^{j-1}}\right]$$

where $\Delta = Q_1(y)\frac{\partial P}{\partial y} + P(y)Q_2(y)$.

Integration over Join cycle

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$\{\delta_\alpha^{-1}\}_{\alpha\in J} \subset H_{n-1}(\{g(x)=-1\})$ basis of vanishing cycles.

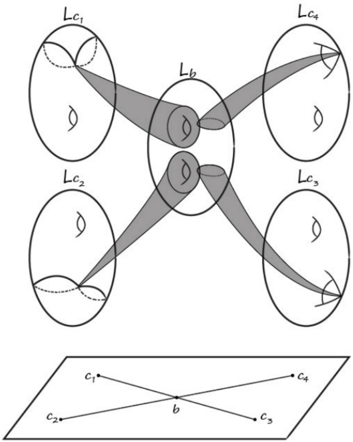

Figure: Vanishing cycles

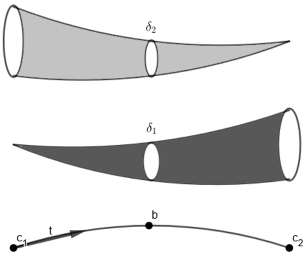

Figure: Join of vanishing cycles

Join cycle

Definition

The cycle

$$\delta_k * \delta_\alpha^{-1} = \delta_1 *_t \delta_\alpha^{-1} := \bigcup_{s\in[0,1]} \delta_{1t_s} \times \delta_{\alpha t_s}^{-1} \in H_n(U,\mathbb{Z})$$

is called the **join cycle** of δ_k and δ_α^{-1} along t.

Theorem ([Mov19], Section 7.9)

$H_n(U,\mathbb{Z})$ is freely generated by

$$\delta_k * \delta_\alpha^{-1}, \ \ k = 0, \cdots, m-2, \ \ \alpha \in J.$$

Where $J = I_{m_1} \times \cdots \times I_{m_n}$ with $I_{m_j} = \{0, 1, 2, \ldots, m_j - 2\}$.

$$“\int_{\delta_1 * \delta_2} = \int_{\delta_1}\int_{\delta_2}”$$

Proposition ([Mov19], Section 13.8)

If $A_{\beta'} \notin \mathbb{N}$, *then*

$$\int_{\delta_k *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{P+g}\right) = \frac{qp(\{g=-1\},\beta',\delta_\alpha^{-1})}{\zeta_q^{\gamma+1}-1}\int_{\delta_k *_t \delta} res\left(\frac{y^{\beta_{n+1}} z^\gamma dy\wedge dz}{P-z^q}\right)$$

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$$\delta = [\zeta_q - 1] \in H_0(\{z^q=1\},\mathbb{Z}).$$

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$$A_{\beta'} := \sum_{j=1}^{n} \frac{\beta_j + 1}{m_j} = \frac{\gamma+1}{q}.$$

$$p(\{g=-1\},\beta',\delta_\alpha^{-1}) := \int_{\delta_\alpha^{-1}} res\left(\frac{x_1^{\beta_1}\dots x_n^{\beta_n} dx_1 \wedge \cdots \wedge dx_n}{g}\right)$$

Example

Consider $f(x,y) = g(x) + P(y)$ where $g = x_1^{m_1} + \cdots + x_n^{m_n}$ and $P(y) := y(1-y)(\lambda - y)$.

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$$\int_{\delta_1 *_t \delta_\alpha^{-1}} res\left(\frac{\omega_\beta}{f^2}\right) = \frac{(\lambda-1)^{2A_{\beta'}-3}p(\{g=-1\},\beta',\delta_\alpha^{-1})}{\lambda^2(\zeta_q^{\gamma+1}-1)}B(A_{\beta'},A_{\beta'})\times$$
$$\left[F\left(A_{\beta'},-(A_{\beta'}+\beta_{n+1}),2A_{\beta'};1-\lambda\right)\times\right.$$
$$\left(3A_{\beta'}+\beta_{n+1}-1\right)a_\lambda+$$
$$F\left(A_{\beta'},-(A_{\beta'}+\beta_{n+1}-1),2A_{\beta'};1-\lambda\right)\times$$
$$((1-A_{\beta'})e_\lambda+(1+\beta_{n+1})b_\lambda)+$$
$$\left.F\left(A_{\beta'},-(A_{\beta'}+\beta_{n+1}-2),2A_{\beta'};1-\lambda\right)\beta_{n+1}c_\lambda\right].$$

Example

$$p(\{g=-1\},\beta',\delta_\alpha^{-1}) = \frac{(-1)^n}{\prod_{j=1}^{n+1} m_j}\prod_{j=1}^{n+1}\left(\zeta_{m_j}^{(\alpha_j+1)(\beta_j+1)} - \zeta_{m_j}^{\alpha_j(\beta_j+1)}\right)\times$$
$$B\left(\frac{\beta_1+1}{m_1},\dots,\frac{\beta_n+1}{m_n},\frac{\beta_{n+1}+1}{m_{n+1}}\right).$$

Example

$$F(a,b,c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n,$$

where $(a)_n$ is the Pochhammer symbol defined by

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$$E(a,b,c): z(1-z)\frac{d^2u}{dz^2}+\{c-(a+b+1)z\}\frac{du}{dz}-abu=0.$$

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$$\mathcal{U} := \{(x, y, P) \in \mathbb{C}^n \times \mathbb{C} \times T | \ g(x) + P(y) = 0\}$$

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The projection $\pi : \mathcal{U} \longrightarrow T$ is a locally trivial C^∞ fibration.

Generic Hodge cycles

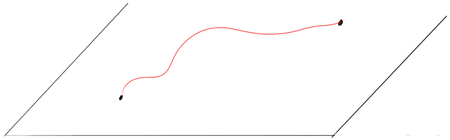
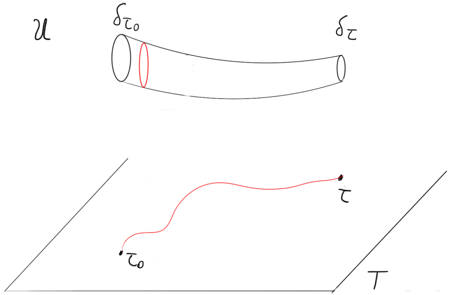


Figure: Generic Hodge cycle.


$\mathcal{U}$
δ_{τ_0}
δ_{τ}
τ
τ_0
T

Definition

Fixed $P_0 \in T$. $\delta_{P_0} \in H_n(\mathcal{U}_{P_0}, \mathbb{Q})$ is called a **generic Hodge cycle** if δ_P is Hodge cycle of X_P, this means, $\delta_P \in Hodge_n(X_P, \mathbb{Q})_0$ for all $P \in T$ and δ_P is monodromy of δ_{P_0} along to a path. We will denote this space by $GHC_n(X_{P_0}, \mathbb{Q})_0$

$\{\delta_k\}_{k=0}^{m-2} \subset H_0(\{P_0(y) = 1\})$ basis of vanishing cycles.

$$\{\delta_k\}_{k=0}^{m-2} \subset H_0(\{P_0(y)=1\}) \text{ basis of vanishing cycles.}$$

$$\{\delta_\alpha^{-1}\}_{\alpha\in J} \subset H_{n-1}(\{g(x)=-1\}) \text{ basis of vanishing cycles.}$$

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$$\delta = \sum_{k=0}^{m-2} \delta^k \text{ with } \delta^k = \sum_{\alpha\in J} n_{k\alpha}\delta_k * \delta_\alpha^{-1}; \qquad n_{k\alpha} \in \mathbb{Q}.$$

Where $J = I_{m_1} \times \cdots \times I_{m_n}$ with $I_{m_j} = \{0,1,2,\ldots,m_j-2\}$.

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$$\int_{\delta} res\left(\frac{\omega_\beta}{f^j}\right) = \frac{1}{\zeta_q^{\gamma+1} - 1} \sum_{k=0}^{m-2} \left(\sum_{\alpha \in J} n_{k\alpha} \int_{\delta_\alpha^{-1}} \frac{\omega_{\beta'}}{dg} \right) \times \int_{\delta_k *_t \hat{\delta}} res\left(\frac{y^{\beta_{n+1}} z^{\gamma} dy \wedge dz}{(P_0 - z^q)^j} \right)$$

Definition (Strong generic Hodge cycles)

Consider the $\mathbb{Q}$-vector space

$$\mathcal{A} := \left\{ (n_{k\alpha}) \in \mathbb{Q}^{|I|} \,\middle|\, \sum_{\alpha \in J} n_{k\alpha} \int_{\delta_\alpha^{-1}} \frac{\omega_{\beta'}}{dg} = 0, \ \forall \beta \text{ s.t. } A_\beta < \frac{n}{2}, k \in I_m \right\}$$

The **strong generic Hodge cycles** is the image of $\mathcal{A}$ by the natural map

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & Hodge_n(X_{P_0}, \mathbb{Q})_0 \\ (n_{k_\alpha}) & \longmapsto & \left[\sum_{k=0}^{m-2} \sum_{\alpha \in J} n_{k\alpha} \delta_k * \delta_\alpha^{-1}\right]. \end{array}$$

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$$= \left\{ (n_{k\alpha}) \in \mathbb{Q}^{|I|} \,\middle|\, \sum_{\alpha \in J} n_{k\alpha} \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ A_{\beta'} < \frac{n}{2} - \frac{1}{m}, k \in I_m \right\}$$

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- $$\mathcal{A} \cong \left\{ (n_\alpha) \in \mathbb{Q}^{|J|} \,\middle|\, \sum_{\alpha \in J} n_\alpha \prod_{j=1}^{n} \zeta_{m_j}^{\alpha_j(\beta_j+1)} = 0,\ A_{\beta'} < \frac{n}{2} - \frac{1}{m} \right\}^{m-1},$$

  and therefore $\delta = \sum_{k=0}^{m-2} \delta^k$ is strong generic Hodge cycle if and only if $\{\delta^k\}_{k=0,\dots,m-2}$ are strong generic Hodge cycle.

Theorem

Let X_{P_0} be desingularization of the weighed hypersurface D given by F homogenization of $f = g(x) + P_0(y)$, where $g(x) = x_1^{m_1} + \cdots + x_n^{m_n}$, $m_i \geq 2$ and $P_0(y)$ is a polynomial of degree m.

1. *For $m_1 = \cdots = m_{n-1} = 2$ and $m \geq 7$, we have*

$$dimSGHC_n(X_{P_0}) \leq \begin{cases} m-1 & m_n \text{ even}, \\ 0 & m_n \text{ odd}. \end{cases}$$

Corollary

In the same context of the theorem with $g(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^{m_n}$ *and* $P_0(y)$ *is a polynomial of degree* 4*. We have*

$$dimSGHC_n(X_{P_0}, \mathbb{Q}) \leq \begin{cases} 0 & m_n \equiv 1, 5, 7, 11 \ (12), \\ 3 & m_n \equiv 2, 10 \ (12), \\ 6 & m_n \equiv 3, 9 \ (12), \\ 9 & m_n \equiv 4, 6, 8 \ (12), \\ 15 & m_n \equiv 0 \ (12). \end{cases}$$

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2. *For $m_1 = \cdots = m_{n-2} = 2$, m_{n-1} prime, $(m_{n-1}, m_n) = 1$ and $\frac{1}{m_{n-1}} + \frac{1}{m} < \frac{1}{2}$, we have $SGHC_n(X_{P_0}, \mathbb{Q}) \cong 0$.*

Corollary

In the same context of the theorem with $g(x) = x_1^2 + x_2^2 + \cdots + x_{n-2}^2 + x_{n-1}^p + x_n^d$, $(p, d) = 1$ *and* $P_0(y)$ *is a polynomial of degree* 3. *We have*

$$dimSGHC_n(X_{P_0}, \mathbb{Q}) \leq \begin{cases} 0 & d \equiv 1, 5\ (6) \\ 4 & d \equiv 2, 4\ (6) \end{cases} \quad for\ p = 3.$$

$$dimSGHC_n(X_{P_0}, \mathbb{Q}) \leq \begin{cases} 0 & d\ \ odd \\ 8 & d\ \ even \end{cases} \quad for\ p = 5.$$

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3. *For m_j different prime numbers, we have $SGHC_n(X_{P_0}, \mathbb{Q}) \cong 0$*

Algebraic values of the hypergeometric function

Proposition (Deligne's [PDS82])

Let X be a smooth projective variety. If $\delta \in H_m(X;\mathbb{Q})$ is algebraic, then for every $\omega \in H^m_{dR}(X/k)$:

$$\frac{1}{(2\pi i)^{m/2}} \int_\delta \omega \in \overline{k},$$

where X/k denotes the variety over a field $k \subset \mathbb{C}$.

Proposition

Let X be desingularization of the weighed hypersurface D given by F homogenization of $f = g(x) + P(y)$, where $g(x) = x_1^2 + x_2^d$ and $P(y) : y(1-y)(\lambda - y)$. Consider $\frac{\omega_\beta}{f}$ a good form with $A_{\beta'} \notin \mathbb{N}$.

Proposition

Let X be desingularization of the weighed hypersurface D given by F homogenization of $f = g(x) + P(y)$, where $g(x) = x_1^2 + x_2^d$ and $P(y) : y(1-y)(\lambda - y)$. Consider $\frac{\omega_\beta}{f}$ a good form with $A_{\beta'} \notin \mathbb{N}$.

$$\delta^0 = \sum_{j=0}^{d-2} n_{0j}\delta_0 * \delta_j^{-1}, \quad \delta^1 = \sum_{j=0}^{d-2} n_{1j}\delta_1 * \delta_j^{-1}.$$

Proposition

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If δ^0 and δ^1 are generic Hodge cycle then it is satisfied that

$$\sum_{j=0}^{d-2} n_{kj}\zeta_d^{j(\beta_2+1)}, \quad k = 0, 1$$

is zero or

$$F\left(A_{\beta'} + \beta_3, 1 - A_{\beta'}, 2A_{\beta'} + \beta_3; \frac{1}{\lambda}\right) \text{ and } F\left(A_{\beta'}, 1 - A_{\beta'} - \beta_3, 2A_{\beta'}; 1 - \lambda\right)$$

are in $\overline{\mathbb{Q}(\lambda)}$.

Idea of the proof:

$$\frac{1}{2\pi i}\int_{\delta} res\left(\frac{\omega_\beta}{f}\right) = \frac{1}{2\pi i}\sum_{j=0}^{d-2} n_j \zeta_d^{j(\beta_2+1)} B\left(\frac{1}{2}, \frac{\beta_2+1}{d}\right) B(A_{\beta'}, A_{\beta'}) F(a,b,c;z)$$

is algebraic over $\overline{\mathbb{Q}(\lambda)}$.

□

Now, as an application we will re-obtain results already showed by Schwarz in [Sch73].

Now, as an application we will re-obtain results already showed by Schwarz in [Sch73].

Corollary

The following expressions are in $\overline{\mathbb{Q}(\lambda)}$

$$F\left(\frac{5}{6},\frac{1}{6}-\beta_3,\frac{5}{3};1-\lambda\right),\quad F\left(\frac{7}{6},\frac{-1}{6}-\beta_3,\frac{7}{3};1-\lambda\right),$$

$$F\left(\frac{5}{6}+\beta_3,\frac{1}{6},\frac{5}{3}+\beta_3;\frac{1}{\lambda}\right),\quad F\left(\frac{7}{6}+\beta_3,\frac{-1}{6},\frac{7}{3}+\beta_3;\frac{1}{\lambda}\right),$$

with $\beta_3=0,1,$

Conjecture

Let X a smooth projective variety. If $\delta \in H_m(X;\mathbb{Q})$ is a Hodge cycle, then for every $\omega \in H^m_{dR}(X/k)$:

$$\frac{1}{(2\pi i)^{m/2}} \int_\delta \omega \in \overline{k},$$

where X/k denotes the variety over a field $k \subset \mathbb{C}$.

Proposition

Let X_n be desingularization of the weighed hypersurface D_n given by F_n homogenization of $f_n = g_n(x) + P(y)$, where $g_n(x) = x_1^2 + \cdots + x_{n-1}^2 + x_n^d$ and $P(y) : y(1-y)(\lambda - y)$. Consider $\frac{\omega_\beta}{f_n}$ a good form with $A_{\beta'} \notin \mathbb{N}$. If $\delta \in H_n(X_n, \mathbb{Q})$ is a strong generic Hodge cycle, we have

$$\frac{1}{(2\pi i)^{n/2}} \int_\delta res\left(\frac{\omega_\beta}{f_n}\right) \in \overline{\mathbb{Q}(\lambda)},$$

Proposition

In the same context of the previous proposition, consider $\frac{\omega_\beta}{f_n^2}$ *a good form such that* $\frac{\omega_\beta}{f_n}$ *is a good form, and* $A_{\beta'} \notin \mathbb{N}$. *If* $\delta \in H_n(X_n, \mathbb{Q})$ *is a strong generic Hodge cycle, we have*

$$\frac{1}{(2\pi i)^{n/2}} \int_\delta res\left(\frac{\omega_\beta}{f_n^2}\right) \in \overline{\mathbb{Q}(\lambda)},$$

Proposition

The following expressions are in $\overline{\mathbb{Q}(\lambda)}$:

$$0 \neq 6F\left(\frac{4}{3},-\frac{4}{3},\frac{8}{3};1-\lambda\right)(\lambda^2-\lambda+1) - \frac{2}{3}F\left(\frac{4}{3},-\frac{1}{3},\frac{8}{3};1-\lambda\right)(\lambda+1)(5\lambda^2-8\lambda+5),$$

$$0 \neq 4F\left(\frac{2}{3},-\frac{5}{3},\frac{4}{3};1-\lambda\right)(\lambda^2-\lambda+1) - \frac{1}{3}F\left(\frac{2}{3},-\frac{2}{3},\frac{4}{3};1-\lambda\right)(\lambda+1)(8\lambda^2-11\lambda+8) + F\left(\frac{2}{3},\frac{1}{3},\frac{4}{3};1-\lambda\right)\lambda(1-\lambda)^2,$$

but each hypergeometric function in the expressions above are not algebraic over $\overline{\mathbb{Q}(\lambda)}$*.*

- Movasati, Hossein: *A course in Hodge theory*, 2019.
  `http://w3.impa.br/~hossein/myarticles/hodgetheory.pdf`.
- P. Deligne, J. S. Milne, Ogus and KY Shih: *Hodge cycles, motives, and Shimura varieties.*
  Lecture Notes in Mathematics, 900, 1982.
- Schwarz, Hermann Amandus: *Ueber diejenigen fälle, in welchen die gaussische hypergeometrische reihe eine algebraische function ihres vierten elementes darstellt.*
  Journal für die reine und angewandte Mathematik, 75:292–335, 1873.


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