On the Picard number and the extension degree of period matrices of complex tori [1]

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Abstract

The rank ρ of the Néron-Severi group of a complex torus X of dimension g satisfies $0 \leq \rho \leq g^2 = h^{1,1}$. The degree $\mathfrak{d}$ of the extension field generated over $\mathbb{Q}$ by the entries of a period matrix of X imposes constraints on its Picard number ρ and, consequently, on the structure of X. In this paper, we show that when $\mathfrak{d}$ is 2, 3 or 4, the Picard number ρ is necessarily large. Moreover, for an abelian variety X of dimension g with $\mathfrak{d} = 3$, we establish a structure-type result: X must be isogenous to either $A^{\frac{g}{2}}$, where A is a simple abelian surface with $\mathrm{End}_{\mathbb{Q}}(A)$ an indefinite quaternion algebra over $\mathbb{Q}$, or E^g, where E is an elliptic curve without complex multiplication. In both cases, the Picard number satisfies $\rho(X) = \frac{g(g+1)}{2}$. As a byproduct, we obtain that if $\mathfrak{d}$ is odd, then $\rho(X) \leq \frac{g(g+1)}{2}$.

1 Introduction

The Picard number ρ is the rank of the Néron-Severi group $\mathrm{NS}(X)$ of a complex manifold X, which parameterizes holomorphic line bundles on X modulo analytic equivalence. When X is a Kähler manifold, the Picard number satisfies

$$\rho \leq h^{1,1} := \dim H^1(X, \Omega^1_X).$$

This number is a fundamental invariant of a variety and, in some cases, determines specific structural properties of the variety. For instance, in the case of K3 surfaces, if $\rho \geq 5$, the surface admits an elliptic fibration and when $\rho \geq 12$, such a fibration exists with a section (see [Huy16, §11.1]). Furthermore, if $\rho \geq 19$, there is an isomorphism of integral Hodge structures between the transcendental lattices of X and some abelian surface [Mor84]. A similar phenomenon occurs for Hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type with large Picard number, see [PM24] for details.

Another more recent example arises in the context of Fano varieties. In [Cas24], it is shown that if X is a smooth Fano 4-fold with Picard number $\rho > 12$, then X must be isomorphic to a product $S_1 \times S_2$, where S_1 and S_2 are del Pezzo surfaces. This result generalizes to dimension 4 the analogous result for Fano 3-folds established in [MM81], which states that if X is a smooth Fano 3-fold with $\rho > 5$, then $X \simeq S \times \mathbb{P}^1$ where S is a del Pezzo surface.

Another notable example arises in the case of abelian varieties, where structure theorems up to isogeny exist for those with large Picard numbers. Specifically, for $g \geq 5$, we have

$$\rho(X) = (g-1)^2 + 1 \iff X \sim E_1^{g-1} \times E_2,$$

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where E_1 has complex multiplication, and E_1 and E_2 are not isogenous. Similarly, for $g \geq 7$, we have

$$\rho(X) = (g-2)^2 + 4 \Longleftrightarrow X \sim E_1^{g-2} \times E_2^2,$$

where E_1 and E_2 both have complex multiplication but are not isogenous. For more details, see [HL19]. In the same paper, a structure result is presented for abelian varieties that achieve the largest possible Picard number among g-dimensional abelian varieties that split into a product of r non-isogenous factors. More precisely, by Poincaré's Complete Reducibility Theorem, any abelian variety X admits a decomposition of the form

$$X \sim A_1^{k_1} \times \cdots \times A_r^{k_r}.$$

Define $r(X) := r$ to be the number of non-isogenous simple factors that appear in the Poincaré decomposition. Now, let

$$M_{r,g} := \max\{\rho(X) | \dim X = g, \ \ r(X) = r\}.$$

In this setting, we have

$$\rho(X) = M_{r(X),g} \Longleftrightarrow X \sim E^{g-r+1} \times E_1 \times \cdots \times E_{r-1},$$

where E is an elliptic curve with complex multiplication that is not isogenous to any of the E_i's, and the curves E_i and E_j are pairwise non-isogenous for $i \neq j$.

If a complex torus X of dimension g satisfies $\rho = h^{1,1} = g^2$, then X must be an abelian variety. Moreover, this occurs if and only if X is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication, which, in turn, happens precisely when the entries of a period matrix of X lie in a quadratic extension of the rational numbers $\mathbb{Q}$. We revisit this last (well-known) fact in Theorem 1.2, and for a complete characterization, see Theorem 3.1. When this condition holds, we say that X is (Picard) ρ-maximal. Complex tori/abelian varieties with this property exhibit interesting arithmetic and geometric properties; see, for instance, [Bea14, SI77, SM74].

Our main theorem, closely related to the two previous examples, is a structural type result (up to isogeny) that further reinforces the phenomenon observed throughout the preceding examples.

Theorem 1.1. Let $\Pi = (\tau \ \ I_g)$ be a period matrix of an abelian variety X of dimension $g \geq 2$, where $\tau = (\tau_{ij})_{g\times g} \in M_g(\mathbb{C})$ satisfies $\det(\mathrm{Im}(\tau)) \neq 0$. Suppose the degree of the field extension generated by the entries of τ over $\mathbb{Q}$ is

$$\mathfrak{d} = [\mathbb{Q}(\{\tau_{ij}\}) : \mathbb{Q}] = 3.$$

Then X is isogenous to a power A^k of a simple abelian variety A. More precisely

1. $X \sim E^g$, where E is an elliptic curve without complex multiplication, or
2. $X \sim A^{\frac{g}{2}}$, where A is a simple abelian surface with $\mathrm{End}_{\mathbb{Q}}(A)$ an indefinite quaternion algebra over $\mathbb{Q}$.

In both cases, the Picard number is given by $\rho(X) = \frac{g(g+1)}{2}$. In particular, when g is odd we have $X \sim E^g$.

Remark 1.1. Theorem 1.1 provides a sufficient elementary condition for an abelian variety X to be isogenous to a power A^k of a simple abelian variety A. In [Wol05, Theorem 9] Wolfart establishes a different set of sufficient conditions in the case of the Jacobian variety Jac C of a curve C. Although his conditions are more sophisticated, they lead to exactly the same two cases as those described in Theorem 1.1.

The proof of Theorem 1.1 relies on the Poincaré Complete Reducibility theorem, the upper bound for the Picard number ρ of a self-product of a simple abelian variety established by Hulek and Laface in [HL19, Corollary 2.5] and the following result, which provides a lower bound for ρ. This theorem provides sufficient conditions for the Picard number ρ of a complex torus X to be large. More precisely:

Theorem 1.2. Let $\Pi = (\tau \;\; I_g)$ be a period matrix of a complex torus X of dimension g, where $\tau = (\tau_{ij})_{g\times g} \in M_g(\mathbb{C})$ satisfies $\det(\operatorname{Im}(\tau)) \neq 0$. Let $\mathfrak{d} = [\mathbb{Q}(\{\tau_{ij}\}) : \mathbb{Q}]$ denote the degree of the field extension generated by the entries of τ over $\mathbb{Q}$. Then we have:

1. $\mathfrak{d} = 2 \Longleftrightarrow X$ is ρ-maximal. In particular, X is an abelian variety.
2. If $\mathfrak{d} = 3$, then $\frac{g(g+1)}{2} \leq \rho < g^2$.
3. If $\mathfrak{d} = 4$, then $g \leq \rho < g^2$.

The proof of Theorem 1.2 relies on an estimate of the dimension of $\ker(T)$ over $\mathbb{Q}$, where T is a certain matrix satisfying $\rho(X) = \dim_{\mathbb{Q}} \ker(T)$; see Proposition 2.1 for further details. Now, if X is an abelian variety and $\mathfrak{d} = 2$ or $\mathfrak{d} = 3$ we have a characterization of X, as given in Theorem 3.1 and Theorem 1.1 respectively. While a similar characterization is not available for $\mathfrak{d} = 4$, by combining Theorem 1.2 and Proposition 3.1 we can further restrict the possible values of the Picard number ρ. By Theorem 3.1 and the additivity of the Picard number, the more powers of elliptic curves with complex multiplication an abelian variety X contains, the larger its Picard number. Proposition 3.1 provides further evidence of this phenomenon. In particular, it implies that if the degree $\mathfrak{d}$ of the field extension over $\mathbb{Q}$ associated with the period matrix of X is odd, then $\rho \leq \frac{g(g+1)}{2}$, see Corollary 3.1.

The structure of this paper is as follows: In Section 2, we introduce the key ingredients for our proofs. Section 3 is dedicated to proving Theorem 1.1 and Theorem 1.2, as well as establishing several interesting consequences.

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2 Preliminaries

In this section, we will show how to calculate the Picard number of a complex torus X in terms of a period matrix for X. This approach allows us to establish lower bounds for the Picard number ρ and, as an application, derive Theorem 1.2. We begin by computing the Picard number ρ in terms of the rank over $\mathbb{Q}$ of a certain matrix T. We recall [BL99, Proposition 1.1] that X has a period matrix of the form $(\tau \;\; I_g)$, where $\det(\operatorname{Im}(\tau)) \neq 0$, and

T will be constructed from the entries of τ. One key advantage of this approach is that, in concrete examples, T can be implemented computationally using only the entries of τ. We will refer to both τ and $(\tau \ \ I_g)$ as period matrices for X, depending on the context.

Proposition 2.1. Let $\Pi = (\tau \ \ I_g)$ be a period matrix of a complex torus X of dimension g, where $\tau = (\tau_{ij}) \in M_g(\mathbb{C})$ satisfies $\det(\mathrm{Im}(\tau)) \neq 0$. Consider the linear map

$$\begin{array}{rcl} T : \mathbb{Q}^{\frac{g(g-1)}{2}} \times \mathbb{Q}^{g^2} \times \mathbb{Q}^{\frac{g(g-1)}{2}} & \longmapsto & \mathbb{C}^{\frac{g(g-1)}{2}} \\ ((a_{ij})_{1\leq i<j\leq g}, (b_{ij}), (c_{ij})_{1\leq i<j\leq g}) & \longmapsto & (w_{ij})_{i<j} \end{array}$$

where

$$w_{ij} = a_{ij} + \sum_{k=1}^{g} (b_{jk}\tau_{ki} - b_{ik}\tau_{kj}) + \sum_{1\leq l<k\leq g} c_{lk}(\tau_{kj}\tau_{li} - \tau_{lj}\tau_{ki}).$$

with $i, j \in \{1, \dots, g\}$. Then the Picard number $\rho = \mathrm{rank}\mathrm{NS}(X)$ satisfies

$$\rho = 2g^2 - g - \mathrm{rank}_{\mathbb{Q}}(T).$$

Proof The Néron Severi group $\mathrm{NS}(X)$ can be identified with the matrix group

$$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \in M_{2g}(\mathbb{R})$$

where A and C are skew-symmetric matrices satisfying the condition

$$W := A - B\tau + \tau^t B^t + \tau^t C\tau = 0,$$

see [BL99, Proposition 1.3.4]. The entries of the skew-symmetric matrix W are explicitly given by

$$w_{ij} = a_{ij} + \sum_{k=1}^{g} (b_{jk}\tau_{ki} - b_{ik}\tau_{kj}) + \sum_{1\leq l<k\leq g} c_{lk}(\tau_{kj}\tau_{li} - \tau_{lj}\tau_{ki})$$

where a_{ij}, b_{ij}, c_{ij} are the entries of the matrices A, B, and C respectively. Given that

$$T((a_{i,j})_{i<j}, (b_{ij}), (c_{ij})_{i<j}) = (w_{ij})_{i<j},$$

we conclude that $\rho = \dim_{\mathbb{Q}} \ker(T)$. Finally, applying the rank-nullity theorem, we obtain the desired result. ∎

Remark 2.1. The matrix associated with the linear map T can be expressed as

$$T = \begin{pmatrix} Id_{\frac{g(g-1)}{2}\times\frac{g(g-1)}{2}} & \mathcal{B}_{\frac{g(g-1)}{2}\times g^2} & \mathcal{C}_{\frac{g(g-1)}{2}\times\frac{g(g-1)}{2}} \end{pmatrix} \in M_{\frac{g(g-1)}{2}\times 2g^2-g}(\mathbb{C})$$

where the matrices $\mathcal{B}$ and $\mathcal{C}$ are defined as follows:

Definition of $\mathcal{B}$: Let $(i_0, j_0) \in \{1, \dots, g\}^2$ be indexed with the lexicographic order, and let $\tau = (\tau_{ij})_{g\times g}$ be the period matrix. The matrix $\mathcal{B} \in M_{\frac{g(g-1)}{2}\times g^2}(\mathbb{C})$ is defined such that its columns (i_0, j_0) are given by

$$\mathcal{B}_{(i,j)(i_0,j_0)} = \begin{cases} \tau_{j_0 i} & i < i_0 \\ -\tau_{j_0 j} & i_0 < j \\ 0 & \text{other cases.} \end{cases}$$

where $i < j \in \{1, \dots, g\}$.

Definition of $\mathcal{C}$: The matrix $\mathcal{C} \in M_{\frac{g(g-1)}{2}d}(\mathbb{C})$ is defined by its entries $\mathcal{C}_{(i,j)(i_0,j_0)}$, for $i < j$ and $i_0 < j_0$, given by of the determinant of the 2×2 submatrix of τ formed by

$$\begin{pmatrix} \tau_{i_0 i} & \tau_{i_0 j} \\ \tau_{j_0 i} & \tau_{j_0 j} \end{pmatrix}.$$

Example 2.1. Using the description of the linear map T from the previous remark, we can directly compute T in the 2-dimensional case. Specifically, we obtain

$$T = (1, -\tau_{12}, -\tau_{22}, \tau_{11}, \tau_{21}, \det(\tau)).$$

Consequently, we recover the well-known formula

$$\rho = 6 - \dim_{\mathbb{Q}} \langle 1, \tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \det \tau \rangle,$$

see [BL99, §2.7].

By loosening the precision of Proposition 2.1, we derive lower bounds for the Picard number ρ of a complex torus. These bounds provide simple criteria for identifying tori with large Picard numbers. We first define one of the fundamental notions of this article:

Definition 2.1. Let $\Pi = (\tau \;\; I_g)$ be a period matrix of a complex torus X of dimension g, where $\tau = (\tau_{ij})_{g \times g} \in M_g(\mathbb{C})$ satisfies $\det(\mathrm{Im}(\tau)) \neq 0$. Define

$$F_X := \mathbb{Q}(\{\tau_{ij}\})$$

$$\mathfrak{d}_X = \mathfrak{d} := [F_X : \mathbb{Q}] \in \mathbb{N} \cup \{\infty\}.$$

Let us take a look at a few properties of F_X:

Proposition 2.2. Let $f : X \to Y$ be a map between complex tori with period matrices $(\tau \;\; I_g)$ and $(\sigma \;\; I_g)$, respectively. Then we have the following:

1. If f has finite kernel, then $F_X \subseteq F_Y$, and in particular $\mathfrak{d}_X \leq \mathfrak{d}_Y$.
2. If f is surjective, then $F_Y \subseteq F_X$, and in particular $\mathfrak{d}_X \geq \mathfrak{d}_Y$.
3. If f is an isogeny, then $F_X = F_Y$, and so $\mathfrak{d}_X = \mathfrak{d}_Y$.
4. $F_X = F_{X^\vee}$.

In particular, 3. implies that F_X, and therefore $\mathfrak{d}_X$, only depends on X and not on the specific period matrix chosen.

Proof Clearly the third item follows from the previous two, and the fourth item is trivial since a period matrix for $X^\vee$ is simply

$$(\tau^\tau \ \ I_g).$$

To prove the first item, the homomorphism $f : X \to Y$ between complex tori induces the following equation:

$$\rho_a(f)(\tau \ \ I_g) = (\sigma \ \ I_g)\,\rho_r(f), \tag{1}$$

where $\rho_a(f)$ is the analytic representation of f and $\rho_r(f)$ is its rational representation. Writing

$$\rho_r(f) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in M_g(\mathbb{Z}),$$

we obtain in particular that

$$\rho_a(f) = \sigma B + D.$$

This implies that each coefficient of the matrix $\rho_a(f)$ belongs to the field F_Y. If f has finite kernel, its analytic representation $\rho_a(f)$ is injective and therefore possesses a left inverse M whose coefficients also lie in F_Y. Using Equation (1) we deduce that

$$\tau = M(\sigma A + C),$$

showing that each coefficient of τ also lies in F_Y. Therefore $F_X \subseteq F_Y$.

Now assume that f is surjective. Then by dualizing, we get an injective morphism $f^\vee : Y^\vee \to X^\vee$, and so by the first and fourth items,

$$F_Y = F_{Y^\vee} \subseteq F_{X^\vee} = F_X.$$

■

Remark 2.2. Let Y be a complex subtorus of X, with period matrices $(\sigma \ \ I_g)$ and $(\tau \ \ I_g)$ respectively. A consequence of Proposition 2.2 is that F_Y is a subfield of F_X, and this only depends on the isogeny classes of Y and X. We will use this fact implicitly throughout the article in various arguments.

We now use the number $\mathfrak{d}$ to obtain certain bounds on the Picard number of a complex torus.

Proposition 2.3. Let $\Pi = (\tau \ \ I_g)$ be a period matrix of a complex torus X of dimension g, where $\tau = (\tau_{ij})_{g\times g} \in M_g(\mathbb{C})$ satisfies $\det(\mathrm{Im}(\tau)) \neq 0$. For each pair (i, j), let

$$\mathfrak{d}_{ij} = \dim_{\mathbb{Q}}\langle 1, \tau_{ki}, \tau_{kj}, \tau_{li}\tau_{kj} - \tau_{ki}\tau_{lj}\rangle_{l,k=1,\dots,g}$$

be the dimension of the $\mathbb{Q}$-vector space generated by these elements. Then the Picard number ρ of X satisfies the inequalities

$$\rho \geq 2g^2 - g - \sum_{1\leq i<j\leq g} \mathfrak{d}_{ij} \geq g^2 - g(g-1)\left(\frac{\mathfrak{d}}{2} - 1\right). \tag{2}$$

Proof According to Proposition 2.1 the Picard number satisfies $\rho = \dim_{\mathbb{Q}} \ker(T)$. A vector $((a_{i,j})_{i<j}, (b_{ij}), (c_{ij})_{i<j}) \in \mathbb{Q}^{2g^2-g}$ belongs to $\ker(T)$ if and only if

$$T((a_{i,j})_{i<j}, (b_{ij}), (c_{ij})_{i<j}) = (w_{ij})_{i<j} = 0,$$

where

$$w_{ij} = a_{ij} + \sum_{k=1}^{g} (b_{jk}\tau_{ki} - b_{ik}\tau_{kj}) + \sum_{1\le l<k\le g} c_{lk}(\tau_{kj}\tau_{li} - \tau_{lj}\tau_{ki}).$$

This system consists of $\frac{g(g-1)}{2}$ equations in $2g^2 - g$ variables, given by $w_{ij} = 0$. The coefficients of each equation $w_{ij} = 0$ belong to the $\mathbb{Q}$-vector space

$$\langle 1, \tau_{ki}, \tau_{kj}, \tau_{li}\tau_{kj} - \tau_{ki}\tau_{lj} \rangle_{l,k=1,\dots,g}.$$

Thus, over $\mathbb{Q}$, the number of independent equations is at most $\sum_{1\le i<j\le g} \mathfrak{d}_{ij}$, with the bound $\mathfrak{d}_{ij} \le \mathfrak{d}$. Therefore, the Picard number ρ is at least the difference between the number of variables and the number of independent equations, yielding Equation (2). ■

We conclude this section with an elementary inequality for positive numbers, which will be used in the proofs of Theorem 1.1 and Proposition 3.1.

Lemma 2.1. Consider the quantity

$$g = \sum_{j=1}^{l} n_j k_j + \sum_{j=l+1}^{r} k_j,$$

where $n_j \ge 2$ and $k_j \ge 1$. Then, the following inequality holds:

$$2\sum_{j=1}^{l} n_j k_j^2 + \sum_{j=l+1}^{r} k_j^2 \le g^2.$$

Moreover, equality holds if and only if

$$l = 1 \wedge n_1 = 2 \wedge r = 0 \text{ or } l = 0 \wedge r = 1,$$

that is, if and only if $g = 2k_1$ or $g = k_1$.

Proof Consider nonnegative integers k_1, k_2. Note that $(k_1+k_2)^2 \ge k_1^2 + k_2^2$, with equality if and only if $k_1 = 0$ or $k_2 = 0$. Using induction, it follows that for $k_j \ge 1$,

$$\sum_{j=l+1}^{r} k_j^2 \le \left(\sum_{j=l+1}^{r} k_j \right)^2, \tag{3}$$

with equality if and only if $l = 0$ and $r = 1$. Taking this into account, for $n_j \ge 2$ and $k_j \ge 1$, we obtain

$$2\sum_{j=1}^{l} n_j k_j^2 \le \sum_{j=1}^{l} n_j^2 k_j^2 \le \left(\sum_{j=1}^{l} n_j k_j \right)^2 \tag{4}$$

with equality if and only if $l = 1$ and $n_1 = 2$. Combining inequalities in Equations (3) and (4) completes the proof. ■

3 Proof of Theorems

In this section, we build upon the results from Section 2 to prove the two main theorems of this article, namely Theorems 1.1 and 1.2.

Recall that a complex torus X is called ρ-maximal if it satisfies $\rho = g^2$. The first part of Theorem 1.2 states that X is ρ-maximal if and only if the field extension associated with its period matrix τ has degree $\mathfrak{d} = 2$. This result was already known in the literature. Here, we provide an elementary proof of one direction of this equivalence and, for completeness, also establish the other direction.

Proof of Theorem 1.2 Let us first prove Item 1. Observe that the inequality given by the extremes in Equation (2) is equivalent to

$$\mathfrak{d} \geq 2\left(1 + \frac{g^2 - \rho}{g(g-1)}\right) \geq 2. \tag{5}$$

This implies that if $\mathfrak{d} = 2$, then $g^2 = \rho$, which occurs if and only if X is ρ-maximal. Reciprocally, if X is ρ-maximal, then by [Bea14, Proposition 3], we have that

$$X \simeq E_1 \times \cdots \times E_g,$$

where the E_i are elliptic curves with complex multiplication, all isogenous to each other. If $\langle \sigma_i, 1\rangle$ is a lattice for E_i, then by Proposition 2.2 we have $\mathbb{Q}(\sigma_i) = \mathbb{Q}(\sigma_j)$ for all i, j. Since the period matrix $(\sigma \;\; I_g)$ of $E_1 \times \cdots \times E_g$ is given by $\sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_g)$, applying Proposition 2.2 again, we obtain

$$F_X = \mathbb{Q}(\sigma_1, \ldots, \sigma_g) = \mathbb{Q}(\sigma_1).$$

Since E_1 has complex multiplication, the field $\mathbb{Q}(\sigma_1)$ has degree 2 over $\mathbb{Q}$, and thus $\mathfrak{d} = 2$. For Items 2. and 3., the lower bounds follow directly from Equation (2), while the upper bound is a consequence of item 1. ■

Remark 3.1. Recall that the algebraic dimension of a complex torus X is defined as the transcendence degree of its field of meromorphic functions $\mathbb{C}(X)$:

$$a(X) := \operatorname{tr}\ \deg_{\mathbb{C}} \mathbb{C}(X).$$

It turns out that if the Picard number satisfies $\rho(X) = 0$, then the algebraic dimension is $a(X) = 0$. A simple yet insightful consequence of Theorem 1.2 or Equation (5) is that a necessary condition for a complex torus X to have Picard number zero is that $\mathfrak{d} \geq 5$. More generally, if the Picard number satisfies $\rho < g$, then we still have $\mathfrak{d} \geq 5$.

The first part of Theorem 1.2 follows the same general principle as [DFVL23, Proposition 2.6], where the authors characterize the Fermat varieties X_d^n whose group of algebraic cycles $H^n(X_d^n, \mathbb{Z})_{\text{alg}}$ attains maximal rank $h^{n/2,n/2}$. This is a natural generalization of the concept of ρ-maximality. By combining the first part of Theorem 1.2 with [Bea14, Proposition 3] we obtain the following characterization of ρ-maximality in complex tori:

Theorem 3.1. Let be X a complex torus of dimension g. We have that

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{End}(X) \leq 2g^2$$

and the following conditions are equivalent

(i) X is ρ-maximal

(ii) $\operatorname{rank}_{\mathbb{Z}} \operatorname{End}(X) = 2g^2$

(iii) X is isogenous to E^g, where E is an elliptic curve with complex multiplication.

(iv) X is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

(v) $\mathfrak{d}_X = 2$.

Remark 3.2. Theorem 3.1 and Theorem 1.1 ilustrates how the Picard number ρ can dictate the structure of a complex torus/abelian variety (see [HL19] for further examples where the Picard number ρ determines the structure of an abelian variety). In particular, Theorem 3.1 implies that if X is ρ-maximal, then X must be an abelian variety, meaning that X is projective. This is a special case of a more general phenomenon: any compact Kähler manifold X that is ρ-maximal is necessarily projective.

Indeed, the Kodaira embedding theorem states that X is projective if and only if there exists some positive class $\eta \in \mathrm{NS}_{\mathbb{Q}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Q})$. In this context, the Picard number ρ is bounded above by the Hodge number $h^{1,1}$, and being p-maximal means that $\rho = h^{1,1}$. Note that the Kähler form ω is already a positive real $(1,1)$-class, ie., $\omega \in H^{1,1}(X) \cap X^2(X, \mathbb{R})$ and is positive. If X is ρ-maximal, then the space $NS_{\mathbb{Q}}$ is dense in $H^{1,1}(X) \cap X^2(X, \mathbb{R})$. This guaranteees the existence of a rational $(1,1)$-class sufficiently close to the Kähler form ω that remains positive. By the Kodaira embedding theorem, this implies that X is projective.

In [HL19, Corollary 2.5], Hulek and Laface established an upper bound for the Picard number ρ of the self-product of a simple abelian variety. Specifically, let A be a simple abelian variety of dimension g and let $k \geq 1$. Then

$$\rho(A^k) \leq \frac{1}{2} gk(2k+1). \tag{6}$$

Using this inequality, we now derive an upper bound for the Picard number ρ of an abelian variety X, which is independent of the field of definition of its period matrix.

Proposition 3.1. Let X be an abelian variety of dimension $g \geq 2$, whose Poincaré decomposition is given by

$$X \sim A_1^{k_1} \times \cdots \times A_l^{k_l} \times E_{l+1}^{k_{l+1}} \times \cdots \times E_r^{k_r},$$

where A_j are simple abelian and E_j are elliptic curves. These factors are pairwise non-isogenous, and the E_j are precisely the elliptic curves with complex multiplication appearing in this decomposition. In this setting, we have the bound

$$\rho(X) \leq \frac{1}{2} \left(g(g+1) + \sum_{j=l+1}^{r} k_i(k_j - 1) \right).$$

In particular, if X does not contain elliptic curves with complex multiplication, its Picard number satisfies $\rho(X) \leq \frac{g(g+1)}{2}$.

Proof Let n_j denote the dimension of A_j, and assume that for some $1 \leq l_1 \leq l$, the abelian varieties A_j are elliptic curves without complex multiplication for $l_1 \leq j \leq l$. Then, the Picard number of the powers of E_j satisfies

$$\rho(E_j^{k_j}) = \frac{k_j(k_j+1)}{2} \text{ for } l_1 \leq j \leq l,$$

and

$$\rho(E_j^{k_j}) = k_j^2 \text{ for } l+1 \leq j \leq r.$$

On the other hand, recall that the Picard number is additive (but not strongly additive), meaning that if A_1 and A_2 are non-isogenous simple abelian varieties, then

$$\rho(A_1 \times A_2) = \rho(A_1) + \rho(A_2),$$

see [HL19, Corollary 2.3]. Thus, using the additivity of the Picard number together with Equation (6), we obtain

$$\begin{aligned}
\rho(X) &\leq \frac{1}{2}\sum_{j=1}^{l_1-1} n_j k_j(2k_j+1) + \frac{1}{2}\sum_{j=l_1}^{l} k_j(k_j+1) + \sum_{j=l+1}^{r} k_j^2 \\
&= \frac{1}{2}\left(g + 2\sum_{j=1}^{l_1-1} n_j k_j^2 + \sum_{j=l_1}^{l} k_j^2 + 2\sum_{j=l+1}^{r} k_j^2 - \sum_{j=l+1}^{r} k_j\right) \\
&\leq \frac{1}{2}\left(g^2 + g + \sum_{j=l}^{r} k_j(k_j-1)\right) \text{ by Lemma 2.1},
\end{aligned}$$

where we have used the identity $g = \sum_{j=1}^{l_1-1} n_j k_j + \sum_{j=l_1}^{r} k_j$. ■

Proposition 3.1 enables us to deduce the following result, which further illustrates how the field over $\mathbb{Q}$ defined by the period matrix of an abelian variety X imposes restrictions on the Picard number ρ.

Corollary 3.1. Let $\Pi = (\tau \ \ I_g)$ be a period matrix of an abelian variety X of dimension g, where $\tau = (\tau_{ij})_{g\times g} \in M_g(\mathbb{C})$ satisfies $\det(\mathrm{Im}(\tau)) \neq 0$. If $\mathfrak{d}$ is odd, then $\rho \leq \frac{g(g+1)}{2}$.

Proof Since $\mathfrak{d}$ is odd, it follows from Proposition 2.2 (see also Remark 2.2) that X does not contain any elliptic curve whose extension field associated is quadratic, i.e., it does not contain any elliptic curve with complex multiplication. Thus, the result follows immediately from Proposition 3.1. ■

In the context of Theorem 1.1, if $\mathfrak{d} = 3$, then using the second part of Theorem 1.2 and Corollary 3.1, we already have $\rho(X) = \frac{g(g+1)}{2}$. Thus, it remains to prove the structure result. To this end, we will use the same idea as Proposition 3.1, but here we will refine the inequalities further. The proof is presented below:

Proof of Theorem 1.1 By Poincaré's Complete Reducibility Theorem [BL04, Thm. 5.3.7], the abelian variety X admits the decomposition

$$X \sim A_1^{k_1} \times \cdots \times A_l^{k_l} \times E_{l+1}^{k_{l+1}} \times \cdots \times E_r^{k_r}, \tag{7}$$

where A_j are simple abelian varieties of dimensions $n_j \geq 2$, E_j are elliptic curves that are pairwise non-isogenous, and $k_j \geq 1$. This decomposition is unique up to isogenies and permutations. Since $\mathfrak{d} = 3$, the extension field over $\mathbb{Q}$ determined by the period matrix of each elliptic curve E_j has also degree 3, by Proposition 2.2. Consequently, E_j does not admit complex multiplication.

On the other hand, recall that the Picard number is additive (but not strongly additive), meaning that if A_1 and A_2 are non-isogenous simple abelian varieties, then

$$\rho(A_1 \times A_2) = \rho(A_1) + \rho(A_2),$$

see [HL19, Corollary 2.3] Using this fact, along with the second part of Theorem 1.2, Equation (6), and the identities

$$g = \sum_{j=1}^{l} n_j k_j + \sum_{j=l+1}^{r} k_j, \quad \rho(E_j^{k_j}) = \frac{k_j(k_j+1)}{2},$$

we obtain the inequality

$$\frac{g(g+1)}{2} \leq \rho(X) = \sum_{j=1}^{l} \rho(A_j^{k_j}) + \sum_{j=l+1}^{r} \rho(E_j^{k_j}) \leq \frac{1}{2}\left(g + 2\sum_{j=1}^{l} n_j k_j^2 + \sum_{j=l+1}^{r} k_j^2\right),$$

from which it follows that

$$g^2 \leq 2\sum_{j=1}^{l} n_j k_j^2 + \sum_{j=1l+1}^{r} k_j^2. \tag{8}$$

By virtue of Lemma 2.1, equality holds in Equation (8). Applying Lemma 2.1 once more, we conclude that the decomposition in Equation (7) must take the form

$$X \sim A^{\frac{g}{2}} \text{ or } X \sim E^g,$$

where A is a simple abelian surface and E is an elliptic curve without complex multiplication.

On the other hand, for an abelian surface A, it is known that $\rho = 3$ if and only if $A \sim E^2$, with E as above, or A is simple with $\mathrm{End}_{\mathbb{Q}}(X)$ being an indefinite quaternion algebra over $\mathbb{Q}$, see [BL99, Proposition 2.7.1]. In our case, A is a simple abelian variety whose associated field extension degree over $\mathbb{Q}$ (determined by the period matrix of A) is 3, implying that $\rho(A) = 3$. This completes the proof. ■

Remark 3.3. From the previous proof, it follows that if an abelian variety X has Picard number $\rho(X) = \frac{g(g+1)}{2}$ and does not contain elliptic curves with complex multiplication then X satisfies the same structure result as Theorem 1.1.

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